A STUDY OF THE HEAT FLOW FOR CLOSED CURVES WITH APPLICATIONS TO GEODESICS

S.K. OTTARSSON

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Mathematics Institute
University of Warwick
COVENTRY CV4 7AL

INTRODUCTION

Each of the two parts of this thesis is a discussion on questions arising from the 1964 paper by J. Eells and J.H. Sampson "Harmonic mappings of Riemannian manifolds" [ES]. To start with, here is a summary of the relevant material from that paper:

Let M (resp. M') be a compact (resp. a complete) Riemannian manifold without boundary. The central problem in [ES] is that of deforming a given mapping $f:M \to M'$ into an extremal of the energy functional E, i.e. into an harmonic mapping. The method used, was that of proving (under certain metric and curvature assumptions on M') the existence of a solution $f_s : M \to M'$, $s \in [0,\infty)$, of the heat equation, which coincides initially (i.e. for $s = 0$) with the given map f, and then proving (under further assumptions on M', in the case where it is non-compact) that such a solution does in fact lead to an extremal. In more detail the proofs go as follows: first, by using a suitable embedding w of M' in some Euclidean space and constructing a Riemannian metric on a Euclidean tubular neighbourhood N of the image such that $w : M' \to N$ becomes an isometric embedding, the harmonic map equation and the heat equation were replaced by global equations (Eqs. ($\bar{1}$) and ($\bar{2}$) p. 140 in [ES], see eqs. (6.1) and (6.2) in Part I, eqs. (6.1) and (6.2) in Part II). Then, using these equivalent global equations, uniqueness and the existence of a solution f_s for small values of the deformation parameter s were proved (Section 10 in [ES]).

The length of the interval of existence was found to depend on the energy density of f and on a compact set containing f (Theorem 10(B) in [ES]). To prove existence for all positive values of the deformation parameter it was therefore necessary to find a priori estimates for the energy density of the solution.

To prove the a priori estimates for the energy density (Theorem 9(B) in [ES]) it was necessary to impose the condition that the target manifold had non-positive Riemannian curvature. To prove the a priori estimates for the second order space derivatives of $w \circ f_s$ (Theorem 9(C) in [ES]) some additional conditions, this time on the embedding w were imposed (conditions (12), Section 8(D) in [ES]). The first of these conditions, i.e. those on the covariant derivative of the projection map $\pi : N \to M'$ because its coefficients appear in the global equations. The second part of (12) ensures that one can use the energy density as a measure on the first order space derivatives of the map $w \circ f_s$. The conditions (12) are automatically satisfied when M' is compact. Under the above-mentioned conditions it was then possible to prove the existence of a unique solution of the heat equation of the kind required, for all positive values of the deformation parameter (Theorem 10(C) in [ES]).

After this result is established, the question remains: will such a solution lead to an extremal of the energy functional? As is pointed out in [ES] (Section 10(D)), for some non-compact manifolds M' there are solutions of the heat equation which are unbounded (i.e. solutions that leave every compact subset of M' for large enough values of the deformation parameter.) Eells and Sampson found a new condition on M', which ensures that this does not happen (Theorem 10(D)). This new condition is again given in terms of the embedding w and the covariant derivative of the projection map π.

Finally, Eells and Sampson proved (Theorem 11A) that given a bounded solution f_s, $s \in [0,\infty)$, of the heat equation, if M' has non-positive Riemannian curvature, then there exists a sequence $s_1, s_2, s_3, \ldots$ of s-values such that the mappings f_{s_k} converge uniformly, along with their first order space derivatives to an harmonic mapping. Here the curvature condition was used both to have the a priori bounds on the space derivatives of the solution and to ensure that the s-derivative of $w \circ f_s$ converged in the mean to zero as $s \to \infty$.

Now for a description of the material presented in this thesis. Throughout it is concerned with closed curves i.e. the domain M above is replaced by the unit circle S^1.

Part I is concerned with the problem of carrying out the method described above in the case of domain S^1, without imposing any curvature restrictions on the target manifold M, and without any conditions depending on an embedding of the manifold. This problem is approached from an angle slightly different from that of Eells and Sampson as follows: first, for a given closed C^1 curve f, a condition on M will be given that will ensure that any solution f_s of the heat equation, which is continuous along with its first S^1-derivative and which coincides with f at s = 0 has its image contained in a fixed compact subset of M. This condition is different from the one in [ES] and does not depend on an embedding. Then, assuming that that is the case (i.e. all solutions uniformly bounded) the existence of a unique solution defined for all positive values of the deformation parameter will be proved. The proof of this result will follow the corresponding proof in [ES] but the conditions on M (in particular the curvature restriction) used there will not be necessary. Finally, in Part I, it is proved that this solution subconverges to a closed geodesic, again following [ES] but without the curvature restriction.

The assertion that the results of [ES] hold for closed curves on compact Riemannian manifolds without curvature restrictions was made in "Variational theory in fibre bundles" by J. Eells and J.H. Sampson, Proc. US-Japan Sem. Diff. Geo. Kyoto (1965) 22-23 and in "On harmonic mappings" by J.H. Sampson, Istituto Nazionale di Alta Matematica Francesco Severi Symposia Matematica Vol. XXVI (1982). The latter contains some remarks about the proof. The proof in this case (i.e. for closed curves on compact Riemannian manifolds of arbitrary curvature) formed the author's M.Sc. dissertation written at Warwick University in the academic year 1981-82. Part I appeared in the Journal of Geometry and Physics Vol. 2, n.1, 1985 under the title "Closed geodesics on Riemannian manifolds via the heat flow".

Part II is taken up with a discussion about the question :

What happens if one considers Lorentz manifolds instead of Riemannian ones? More specifically, given a closed curve f on a Lorentz manifold M, can one prove results about existence, convergence etc. of a solution f_s of the heat equation with $f_0 = f$, of the same kind as those in [ES]? The change from a positive definite metric to a non-definite one turned out to change the properties of the solutions of the heat equation as is illustrated by some examples in Section 5 of Part II. For instance, there are examples of solutions for which the energy is not bounded and examples of solutions which only exist up to a finite value of the deformation parameter. However, there are also examples of solutions which exist for all $s \geq 0$, are bounded along with their derivatives and which converge uniformly to a closed geodesic. Part II is then devoted to the investigation of to what extent one can carry out the method from [ES] described above. In more detail as follows: The idea that one could apply the results already obtained in Part I lead to the study of certain Riemannian metrics naturally associated to time-orientable Lorentz manifolds. However, the result was that this approach only works in the special case where the Lorentz manifold has a parallel timelike vector field. As to the causal character of curves, it turned out that applying heat flow to closed timelike curves, the solution will preserve that property, i.e. will define a t-homotopy of the initial curve. The property of being a spacelike or lightlike closed curve is, however, in general, not preserved.

The simple property of solutions on Riemannian manifolds that the energy of the curves f_s decreases as s increases does not have an analogue on Lorentz manifolds (see examples III and IV of Section 5, Part II). For timelike solutions however, one can say a bit more both about the evolution of the energy and of the "length" as defined by the Lorentz metric, in fact, the length of the curves f_s for such solutions increases as s increases.

As for the existence of solutions, the main difficulty lies in the fact that one cannot use the energy density, in the same way as in [ES] and Part I, as a measure on the first derivative of the curves f_s, both because the energy density is, in general, not bounded, and even if it were, that would, because the metric is non-definite, not imply boundedness in any Riemannian metric. The proofs of uniqueness and of existence for small values of the deformation parameter are the same as for Riemannian metrics, but in order to prove existence for all positive values it was necessary to make certain new boundedness assumptions on the first θ-derivative of the solution.

Finally, in Part II there is a proof of subconvergence of bounded timelike solutions with bounded first θ-derivative to a closed geodesic.

This thesis has benefitted from the valuable advice and guidance of Professor J. Eells.

PART I

1. CONTENTS

The material of Part I is arranged in sections as follows: the notation used is fixed in Section 2 which also contains some basic definitions and results in differential geometry. The definitions of energy, tension field etc. are given in Section 3, along with some fundamental properties of solutions of the heat equation. The condition for boundedness of solutions is given in Section 4 and Section 5 has some results about when, in terms of the geometry of the manifold, such a condition might be fulfilled. The proof of existence for all positive values of the deformation parameter is in Section 6 and the proof of sub-convergence of the solution to a closed geodesic is in Section 7.

2. NOTATION

Throughout M will denote a complete Riemannian manifold. $\langle u,v\rangle$ is the inner product of two tangent vectors u, v at the same point on M and $|v| = \langle v,v\rangle^{\frac{1}{2}}$ the length of v. d is the distance function on M and $B_r(x)$ the open ball centred at x with radius r.

The unit circle S^1 will always be parametrized by the central angle θ. If f is a mapping with domain $S^1 \times I$ where I is a subset of $\mathbb{R}$, for a fixed t in I f_t is the mapping with domain S^1 given by $f_t(\theta) = f(\theta,t)$. $f:S^1 \to \mathbb{R}$ will sometimes be identified with $f \circ (\theta \to e^{i\theta}):\mathbb{R} \to \mathbb{R}$. ∂_θ denotes differentiation with respect to θ, $\partial_\theta f = f_* \partial_\theta$ where f_* is the differential of f, similarly for ∂_t.

The symbol ∇ will be used for the Levi-Civita connection on M and for the induced connection on the vector fields along a smooth mapping f into M. The following facts about the connection (from [GKM]) will be needed.

Let X, Y, Z be vector fields on M. The torsion tensor T defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{2.1}$$

satisfies $T \equiv 0$. The curvature tensor R is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Let N be a smooth manifold, $f:N \to M$ smooth, A, B vector fields on N and X, Y vector fields along f. Then

$$0 = T(f_*A, f_*B) = \nabla_A f_*B - \nabla_B f_*A - f_*[A,B] \tag{2.2}$$

$$R(f_*A, f_*B)X = \nabla_A \nabla_B X - \nabla_B \nabla_A X - \nabla_{[A,B]} X \tag{2.3}$$

$$A\langle X,Y\rangle = \langle \nabla_A X, Y\rangle + \langle X, \nabla_A Y\rangle \tag{2.4}$$

3. THE HEAT EQUATION

Definitions:

For a C^1 curve $f:S^1 \to M$ its energy density $e(f)$ is the function $S^1 \to \mathbb{R}$ defined by $e(f)(\theta) = \frac{1}{2}|\partial_\theta f(\theta)|^2$. The energy of f $E(f)$ is defined by $E(f) = \int_0^{2\pi} e(f)(\theta)d\theta$. The length of a C^1 curve g $L(g)$ is the integral of the length of its tangent vector over its domain. When f is C^2 its tension field $\tau(f)$ is the vector field along f given by $\tau(f) = \nabla_{\partial_\theta}\partial_\theta f$. f is a geodesic iff its tension field vanishes.

Lemma 3A:

Let $f_t:S^1 \to M$ be a smooth family of closed curves for t in some open interval. Put $E(t) = E(f_t)$. Then

$$\partial_t E(t) = -\int_0^{2\pi} \langle\tau(f_t)(\theta), \partial_t f_t(\theta)\rangle d\theta.$$

Proof:

By definition

$$E(t) = \int_0^{2\pi} \frac{1}{2}\langle\partial_\theta f_t(\theta), \partial_\theta f_t(\theta)\rangle d\theta$$

so by (2.4) and (2.2)

$$\partial_t E(t) = \int_0^{2\pi} \langle\nabla_{\partial_t}\partial_\theta f_t(\theta), \partial_\theta f_t(\theta)\rangle d\theta = \int_0^{2\pi} \langle\nabla_{\partial_\theta}\partial_t f_t(\theta), \partial_\theta f_t(\theta)\rangle d\theta.$$

Further

$$0 = \int_0^{2\pi} \partial_\theta \langle \partial_t f_t(\theta), \partial_\theta f_t(\theta) \rangle d\theta = \int_0^{2\pi} [\langle \nabla_{\partial_\theta} \partial_t f_t(\theta), \partial_\theta f_t(\theta) \rangle$$

$$+ \langle \partial_t f_t(\theta), \nabla_{\partial_\theta} \partial_\theta f_t(\theta) \rangle] d\theta,$$

and from this

$$\int_0^{2\pi} \langle \nabla_{\partial_\theta} \partial_t f_t(\theta), \partial_\theta f_t(\theta) \rangle d\theta = - \int_0^{2\pi} \langle \nabla_{\partial_\theta} \partial_\theta f_t(\theta), \partial_t f_t(\theta) \rangle d\theta . \square$$

Corollary:

If f_t is a smooth solution of the heat equation

$$\partial_t f_t = \tau(f_t) \tag{3.1}$$

then

$$\partial_t E(t) = - \int_0^{2\pi} |\partial_t f_t|^2 d\theta . \tag{3.2}$$

Therefore $\partial_t E(t) \leq 0$ with $\partial_t E(t) = 0$ only when f_t is a geodesic.

The following propositions are special cases of Propositions 2(B) and 6(B) of [ES].

Proposition:

Every C^2 curve $f: S^1 \to M$ which satisfies $\tau(f) = 0$ is smooth.

Proposition:

If $(\theta,t) \to f_t(\theta)$ is a map of $S^1 \times (t_0,t_1) \to M$ which is C^1 on the product manifold and C^2 on S^1 for each t, and if that map satisfies (3.1), then it is smooth.

4. BOUNDEDNESS CONDITIONS

Solutions of the heat equation that are not bounded exist as is shown by the following example from [ES].

Let N be the manifold obtained by revolving the graph of $v(u) = 1 + e^{-u}$ around the u-axis. If f satisfies $\partial_\theta u = 0$, $\phi = \theta$ (ϕ revolution angle) then so does the solution f_t for any subsequent time. The heat equation reduces to $\partial_t u = \frac{e^u+1}{e^{2u}+1}$. Thus $e^u + u - 2\log(e^u+1) = t + \text{const}$, in particular $u \to \infty$ as $t \to \infty$. However, if the length of f is less than 2π one would expect the solution to be bounded.

Fix a C^1 curve $f:S^1 \to M$. Throughout this section $f_t:S^1 \to M$ is a solution of the heat equation for $0 \leq t < b \leq \infty$ which is continuous along with $\partial_\theta f_t$ at $t = 0$ and which coincides with f at $t = 0$. As before $E(t) = E(f_t)$.

<u>Definition</u>:

Let $c > 0$ and U be an open set in M. U has the property P(c) if for every C^2 curve $g:S^1 \to U$

$$cE(g) \leq \int_0^{2\pi} |\tau(g)|^2 d\theta \tag{4.1}$$

Put $m(f,c) = (2\pi)^{-\frac{1}{2}}E(f) + (3\pi^{\frac{1}{2}} + 2(2\pi)^{-\frac{1}{2}})E(f)^{\frac{1}{2}} + 4(2\pi)^{-\frac{1}{2}}c^{-1}E(f)^{1/4}$.

<u>Theorem 4A</u>:

Let M satisfy the following condition: There is a compact $K \subset M$ such that for every x in the complement of K there exist real numbers

k,c with $c > 0$ and $k > m(f,c)$ such that $B_k(x)$ has the property $P(c)$. Then the image of f_t is bounded, and further if $b = \infty$ and if the image of some f_{t_0} lies in $M \setminus K$ then there exists a $p \in M$ such that $\lim_{t \to \infty} f_t \equiv p$.

<u>Proof</u>:

For every closed C^1 curve $g:S^1 \to M$ one has the inequalities

$$\operatorname{diam} g(S^1) \leq \frac{1}{2} L(g) \leq \pi^{\frac{1}{2}} E(g)^{\frac{1}{2}} \tag{4.2}$$

By the corollary to Lemma 3A $E(t)$ is a non-increasing function and is therefore bounded by $E(f)$. Bearing that and (4.2) in mind it is seen that if all the curves f_t with $t \in (o,b)$ intersect K then the image of the solution is bounded.

So suppose that the image of f_{t_0} lies in $M \setminus K$ for some $t_0 \in (0,b)$. Fix $\theta_0 \in S^1$. By hypothesis there exist $c > 0$ and $k > m(f,c)$ such that $B_k(f_{t_0}(\theta_0))$ has the property $P(c)$. It is easily seen that $f_{t_0}(S^1) \subset B_k(f_{t_0}(\theta_0))$. Put $t_1 = \sup\{t' > t_0 : f_t(S^1) \subset B_k(f_{t_0}(\theta_0))$ for all $t \in [t_0,t')\}$ and suppose for a contradiction that $t_1 < b$. It is easy to show that

$$\sup_{\theta \in S^1} d(f_{t_1}(\theta), f_{t_0}(\theta)) \leq \pi^{\frac{1}{2}}[E(t_0)^{\frac{1}{2}} + E(t_1)^{\frac{1}{2}}] + \frac{1}{2\pi}\int_0^{2\pi}\int_{t_0}^{t_1} |\partial_t f_t(\theta)| dt\, d\theta. \tag{4.3}$$

Estimate of the last term in (4.3): By (3.2)

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{t_0}^{t_1}|\partial_t f_t(\theta)|\,dt\,d\theta \le (2\pi)^{-\frac{1}{2}}\int_{t_0}^{t_1}\left(\int_0^{2\pi}|\partial_t f_t(\theta)|^2 d\theta\right)^{\frac{1}{2}} dt =$$

$$= (2\pi)^{-\frac{1}{2}}\int_{t_0}^{t_1}(-\partial_t E(t))^{\frac{1}{2}}\,dt. \qquad (4.4)$$

Put $T_1 = \{t \ge t_0 : -\partial_t E(t) \ge 1\}$, $T_2 = \{t \ge t_0 : E(t)^{\frac{1}{2}} \le -\partial_t E(t) \le 1\}$, $T_3 = \{t \ge t_0 : -\partial_t E(t) \le E(t)^{\frac{1}{2}}\}$. From (4.4)

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{t_0}^{t_1}|\partial_t f_t(\theta)|\,dt\,d\theta \le (2\pi)^{-\frac{1}{2}}\int_{[t_0,t_1]\cap T_1}(-\partial_t E(t))^{\frac{1}{2}}dt +$$

$$+ (2\pi)^{-\frac{1}{2}}\int_{[t_0,t_1]\cap T_2}(-\partial_t E(t))^{\frac{1}{2}}dt + (2\pi)^{-\frac{1}{2}}\int_{[t_0,t_1]\cap T_3}(-\partial_t E(t))^{\frac{1}{2}}dt \qquad (4.5)$$

I $\quad (2\pi)^{-\frac{1}{2}}\int_{[t_0,t_1]\cap T_1}(-\partial_t E(t))^{\frac{1}{2}}dt \le (2\pi)^{-\frac{1}{2}}\int_{[t_0,t_1]\cap T_1}(-\partial_t E(t))\,dt \le$

$$\le (2\pi)^{-\frac{1}{2}}(E(t_0) - E(t_1)) \qquad (4.6)$$

II $\quad$ Suppose $[t',t''] \subset T_2$ i.e. $E(t)^{\frac{1}{2}} \le -\partial_t E(t) \le 1$ for all $t \in [t',t'']$.
Then

$$1 \le -\frac{\partial_t E(t)}{E(t)^{\frac{1}{2}}} = -2\partial_t(E^{\frac{1}{2}}) \quad \text{so}$$

$$t'' - t' \le -2\int_{t'}^{t''}\partial_t(E^{\frac{1}{2}})\,dt = 2(E(t')^{\frac{1}{2}} - E(t'')^{\frac{1}{2}}).$$

Therefore, the sum of the lengths of the intervals making up $[t_o,t_1] \cap T_2$ is no greater than $2(E(t_o)^{\frac{1}{2}} - E(t_1)^{\frac{1}{2}})$ and so

$$(2\pi)^{-\frac{1}{2}} \int_{[t_o,t_1]\cap T_2} (-\partial_t E(t))^{\frac{1}{2}} dt \leq 2(2\pi)^{-\frac{1}{2}}(E(t_o)^{\frac{1}{2}} - E(t_1)^{\frac{1}{2}}) \tag{4.7}$$

III Since $B_k(f_{t_o}(\theta_o))$ has the property $P(c)$ one has

$$cE(t) \leq \int_0^{2\pi} |\tau(f_t)(\theta)|^2 d\theta = \int_0^{2\pi} |\partial_t f_t(\theta)|^2 d\theta = -\partial_t E(t)$$

and therefore

$$\frac{\partial_t E(t)}{E(t)} \leq -c, \text{ for all } t \in [t_o,t_1).$$

From this it follows that for $t \in [t_o,t_1)$

$$E(t) \leq E(t_o)e^{-c(t-t_o)}. \tag{4.8}$$

Therefore

$$(2\pi)^{-\frac{1}{2}} \int_{[t_o,t_1]\cap T_3} (-\partial_t E(t))^{\frac{1}{2}} dt \leq (2\pi)^{-\frac{1}{2}} \int_{[t_o,t_1]\cap T_3} E(t)^{1/4} dt \leq$$

$$\leq (2\pi)^{-\frac{1}{2}} \int_{t_o}^{t_1} E(t_o)^{1/4} \exp[\frac{-c(t-t_o)}{4}] dt = 4(2\pi)^{-\frac{1}{2}} c^{-1} E(t_o)^{1/4}(1-\exp[\frac{-c(t_1-t_o)}{4}]) \tag{4.9}$$

Combining (4.3), (4.5), (4.6), (4.7), (4.9)

$$\sup_{\theta\in S^1} d(f_{t_1}(\theta),f_{t_0}(\theta)) \le \pi^{\frac{1}{2}}[E(t_0)^{\frac{1}{2}} + E(t_1)^{\frac{1}{2}}] + (2\pi)^{-\frac{1}{2}}[E(t_0) - E(t_1)] +$$

$$+ 2(2\pi)^{-\frac{1}{2}}[E(t_0)^{\frac{1}{2}} - E(t_1)^{\frac{1}{2}}] + 4(2\pi)^{-\frac{1}{2}}c^{-1}E(t_0)^{1/4}(1-\exp[\frac{-c(t_1-t_0)}{4}]). \quad (4.10)$$

And further

$$\sup_{\theta\in S^1} d(f_{t_1}(\theta),f_{t_0}(\theta)) \le (2\pi)^{-\frac{1}{2}}E(t_0)+(2\pi^{\frac{1}{2}}+2(2\pi)^{-\frac{1}{2}})E(t_0)^{\frac{1}{2}}+4(2\pi)^{-\frac{1}{2}}c^{-1}E(t_0)^{1/4}.$$

From this it follows that $f_{t_1}(S^1) \subset B_k(f_{t_0}(\theta_0))$, but that would mean that there is an $\varepsilon > 0$ such that $f_{t_1+\tau}(S^1) \subset B_k(f_{t_0}(\theta_0))$ for all $\tau \in [0,\varepsilon)$ contradicting the choice of t_1.

For the proof of the last assertion of the theorem observe that by the above proof $f_t(S^1) \subset B_k(f_{t_0}(\theta_0))$ for all $t \in [t_0,\infty)$ and therefore the inequality (4.8) will hold for all $t \in [t_0,\infty)$. Further, for all $t',t'' \in [t_0,\infty)$ the inequality (4.10) will hold for t', t'' in place of t_0, t_1. For a fixed θ (4.8) and (4.10) show that $f_t(\theta)$ converges to a point p as $t \to \infty$. Further, (4.8) shows that $E(t)$ and therefore (by (4.2)) $L(f_t)$ converge to zero, which shows that $\lim_{t\to\infty} f(\theta)$ is independent of θ.

Remark:

In the case of $M = \mathbb{R}^n$ it is easy to show that the inequality $E(f) \le \int_0^{2\pi} |\tau(f)|^2 d\theta$ holds for every closed C^2 curve $f:S^1 \to \mathbb{R}^n$:

First consider real valued f. For any $f \in L^1(S^1)$ the Fourier coefficients of f are defined by the formula

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta \qquad n \in \mathbb{Z}$$

and one has the Parseval theorem

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}d\theta$$

whenever $f,\ g \in L^2(S^1)$; the series on the left converges absolutely. When f is C^2 one has

$$\widehat{\partial_\theta f}(n) = in\hat{f}(n),\ \widehat{\partial_\theta^2 f}(n) = -n^2\hat{f}(n) \qquad n \in \mathbb{Z}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\partial_\theta f(\theta)|^2 d\theta = \sum_{n=-\infty}^{\infty} |\widehat{\partial_\theta f}(n)|^2 = \sum_{n=-\infty}^{\infty} n^2|\hat{f}(n)|^2$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\partial_\theta^2 f(\theta)|^2 d\theta = \sum_{n=-\infty}^{\infty} n^4|\hat{f}(n)|^2.$$

Therefore $\|\partial_\theta f\|_2^2 \le \|\partial_\theta^2 f\|_2^2$ and the same inequality holds for $\mathbb{R}^n$-valued f because it holds for each component.

5. MORE ON BOUNDEDNESS CONDITIONS

In this section a demonstration of the inequality

$$\frac{1}{2\pi^2} E(f) \leq \int_0^{2\pi} |\tau(f)|^2 d\theta$$

for closed curves satisfying certain assumptions, following a proof of the Synge-formula given on pp. 122-3 in [GKM].

Definition:

If A is a subset of M define

$$\kappa_A = \sup \{\langle R(X,Y)Y,X\rangle : X,Y \in T_pM,\ |X| = |Y| = 1,\ p \in A\}.$$

Then for all $p \in A$, $X,Y \in T_pM$, $\langle R(X,Y)Y,X\rangle \leq \kappa_A |X|^2 |Y|^2$.

Theorem 5A:

Let $f:S^1 \to M$ be a non-constant closed C^2 curve and assume w.l.o.g. that the maximum of the energy density occurs at $\theta = 0$. Suppose that the image of f is contained in a ball $B_r(f(0))$ satisfying the following: The exponential map $\exp_{f(0)}:T_{f(0)}M \to M$ restricted to $B_r(0)$ in $T_{f(0)}M$ is a diffeomorphism onto $B_r(f(0))$ and if $\kappa_B > 0$ the radius r satisfies $r < (2\kappa_B^{1/2})^{-1}$. Then the inequality, $\frac{1}{2\pi^2} E(f) \leq \int_0^{2\pi} |\tau(f)|^2 d\theta$ holds.

Remark:

The inequality is of course trivial when f is constant.

During the proof consider θ in the interval $(0,\theta_0)$ where θ_0 is the first $\theta > 0$ such that $f(\theta) = f(0)$. Put $B = B_r(f(0))$. To begin with a few definitions.

Define $f^*: S^1 \to T_{f(0)}M$ by $f^* = \exp^{-1}_{f(0)} \circ f$ and a family of geodesics g by $g(\theta,t) = \exp_{f(0)}(t.f^*(\theta))$. For a fixed θ $g^\theta(\cdot) = g(\theta,\cdot)$ is a geodesic from $f(0)$ to $f(\theta)$ ($g^\theta(0) = f(0)$, $g^\theta(1) = f(\theta)$).

Define three vector fields along the map g: $X = \partial_t g^\theta$, $Y = \partial_\theta g^\theta$ and $\tilde{Y} = Y - \langle Y,X\rangle X / |X|^2$.

Define $L(\theta)$ to be the length of g^θ. $L(\theta) = |\partial_t g^\theta| = |X|$.

<u>Step I</u>:

$\lim_{\theta\to 0^+} X/|X|(\theta,t)$ exists and equals the unit vector in the direction of $\partial_\theta f|_{\theta=0}$.

<u>Proof</u>:

Identify $T_{f(0)}M$ and all its tangent spaces with $\mathbb{R}^n$ and so let $J_v(u)$ be the vector u translated to $(T_{f(0)}M)_v$. One has

$$\frac{X}{|X|}(\theta,t) = \exp_{f(0)*}\left(J_{tf^*(\theta)}\frac{f^*(\theta)}{|f^*(\theta)|}\right)$$

(compare Gauß-lemma p. 136 [GKM]). $\lim_{\theta\to 0^+}\frac{f^*(\theta)}{|f^*(\theta)|}$ exists (in $T_{f(0)}M$) and equals the unit vector in the direction of $\partial_\theta f|_{\theta=0}$ and so $\lim_{\theta\to 0^+}\frac{X}{|X|}(\theta,t)$ exists and is equal to the same vector. □

$$\left.\begin{array}{l}\text{Since } g(\theta,0) \equiv f(0),\ Y(\theta,0) \equiv 0 \text{ and also } \nabla_{\partial_\theta} Y(\theta,0) \equiv 0 \\ \text{Since } g(\theta,1) = f(\theta),\ Y(\theta,1) = \partial_\theta f(\theta) \text{ and also } \nabla_{\partial_\theta} Y(\theta,1) = \tau(f)(\theta)\end{array}\right\} \quad (5.1)$$

The vector field $\tilde{Y}$ along g (the component of Y orthogonal to X) is C^1 for $\theta \in (0,\theta_0)$ and $\tilde{Y}(\theta,0) \equiv 0$ by (5.1).

Step II:

$$\partial_\theta L(\theta) = \langle \partial_\theta f(\theta), \frac{X}{|X|}(\theta,1)\rangle \tag{5.2}$$

Proof:

By (2.2) $\quad \nabla_{\partial_\theta} X = \nabla_{\partial_t} Y .$ $\qquad$ (5.3)

Using this and $L(\theta) = \int_0^1 |X|\, dt$ there follows

$$\partial_\theta L(\theta) = \int_0^1 \frac{\partial_\theta \langle X,X\rangle}{2|X|}\, dt = \int_0^1 \frac{\langle \nabla_{\partial_\theta} X, X\rangle}{|X|}\, dt = \int_0^1 \frac{\langle \nabla_{\partial_t} Y, X\rangle}{|X|}\, dt = \int_0^1 \partial_t \langle Y, \frac{X}{|X|}\rangle dt =$$

$$= \langle Y, \frac{X}{|X|}\rangle(\theta,t)\Big|_{t=0,1} = \langle \partial_\theta f(\theta), \frac{X}{|X|}(\theta,1)\rangle \text{ by (5.1).} \quad \square$$

Corollary:

$$\lim_{\theta\to 0^+} \partial_\theta L(\theta) = |\partial_\theta f(0)|. \tag{5.4}$$

This follows immediately from (5.2) and Step I. $\square$

Step III:

$$\partial_\theta^2 L(\theta) = \int_0^1 \frac{1}{|X|} (\langle \nabla_{\partial_t} \tilde{Y}, \nabla_{\partial_t} \tilde{Y}\rangle - \langle R(X,\tilde{Y})\tilde{Y}, X\rangle)\, dt +$$

$$+ \langle \tau(f)(\theta), \frac{X}{|X|}(\theta,1)\rangle \tag{5.5}$$

Proof:

By (5.3) $\partial_\theta(\frac{1}{|X|}) = \partial_\theta\langle X,X\rangle^{-\frac{1}{2}} = -\langle X,X\rangle^{-3/2}\langle\nabla_{\partial_\theta}X,X\rangle =$

$$= -\frac{1}{|X|^3}\langle\nabla_{\partial_t}Y,X\rangle \tag{5.6}$$

From above $\partial_\theta L(\theta) = \int_0^1 \frac{1}{|X|}\langle\nabla_{\partial_t}Y,X\rangle dt$, therefore using (5.6) and (5.3) one obtains

$$\partial_\theta^2 L(\theta) = \int_0^1 (\frac{1}{|X|}(\partial_\theta\langle\nabla_{\partial_t}Y,X\rangle) + (\partial_\theta\frac{1}{|X|})\langle\nabla_{\partial_t}Y,X\rangle)dt =$$

$$= \int_0^1 (\frac{1}{|X|}(\langle\nabla_{\partial_\theta}\nabla_{\partial_t}Y,X\rangle + \langle\nabla_{\partial_t}Y,\nabla_{\partial_\theta}X\rangle) - \frac{1}{|X|^3}\langle\nabla_{\partial_t}Y,X\rangle^2)dt =$$

$$= \int_0^1 (\frac{1}{|X|}(\langle\nabla_{\partial_\theta}\nabla_{\partial_t}Y,X\rangle + \langle\nabla_{\partial_t}Y,\nabla_{\partial_t}Y\rangle) - \frac{1}{|X|^3}\langle\nabla_{\partial_t}Y,X\rangle^2)dt. \tag{5.7}$$

Since for a fixed θ $g(\theta,\cdot)$ is a geodesic $\nabla_{\partial_t}X = 0$ and $\partial_t|X| = 0$. Therefore

$$\nabla_{\partial_t}(\frac{1}{|X|^2}\langle Y,X\rangle X) = \frac{1}{|X|^2}(\partial_t\langle Y,X\rangle)X \tag{5.8}$$

Also, since $\langle\tilde{Y},X\rangle = 0$

$$\langle\nabla_{\partial_t}\tilde{Y},X\rangle = \partial_t\langle\tilde{Y},X\rangle = 0 \tag{5.9}$$

Now $Y = \tilde{Y} + \frac{1}{|X|^2}\langle Y,X\rangle X$ and $\nabla_{\partial_t}Y = \nabla_{\partial_t}\tilde{Y} + \nabla_{\partial_t}(\frac{1}{|X|^2}\langle Y,X\rangle X) =$

$= \nabla_{\partial_t}\tilde{Y} + \frac{1}{|X|^2}(\partial_t\langle Y,X\rangle)X$ by (5.8). From this it follows that

$$\langle \nabla_{\partial_t} Y, \nabla_{\partial_t} Y \rangle = \langle \nabla_{\partial_t} \tilde{Y}, \nabla_{\partial_t} \tilde{Y} \rangle + \frac{1}{|X|^2} (\partial_t \langle Y,X \rangle).2\langle \nabla_{\partial_t} \tilde{Y}, X \rangle +$$

$$+ \frac{1}{|X|^4} (\partial_t \langle Y,X \rangle)^2 \langle X,X \rangle.$$

The second term in this expression vanishes by (5.9), the last one equals $\frac{1}{|X|^2}(\langle \nabla_{\partial_t} Y,X \rangle)^2$. In other words

$$\langle \nabla_{\partial_t} Y, \nabla_{\partial_t} Y \rangle = \langle \nabla_{\partial_t} \tilde{Y}, \nabla_{\partial_t} \tilde{Y} \rangle) + \frac{1}{|X|^2} (\langle \nabla_{\partial_t} Y,X \rangle)^2.$$

Substituting this in (5.7) gives

$$\partial_\theta^2 L(\theta) = \int_0^1 \frac{1}{|X|} (\langle \nabla_{\partial_\theta} \nabla_{\partial_t} Y,X \rangle + \langle \nabla_{\partial_t} \tilde{Y}, \nabla_{\partial_t} \tilde{Y} \rangle)dt \tag{5.10}$$

By (2.3) $\nabla_{\partial_\theta} \nabla_{\partial_t} Y = \nabla_{\partial_t} \nabla_{\partial_\theta} Y - R(X,Y)Y$. Therefore $(\nabla_{\partial_\theta} \nabla_{\partial_t} Y,X \rangle =$

$$= \langle \nabla_{\partial_t} \nabla_{\partial_\theta} Y,X \rangle - \langle R(X,Y)Y,X \rangle = \partial_t (\nabla_{\partial_\theta} Y,X \rangle - \langle R(X,Y)Y,X \rangle.$$

Substituting this in (5.10) gives

$$\partial_\theta^2 L(\theta) = \int_0^1 (\partial_t \langle \nabla_{\partial_\theta} Y, \frac{X}{|X|} \rangle + \frac{1}{|X|}(\langle \nabla_{\partial_t} \tilde{Y}, \nabla_{\partial_t} \tilde{Y} \rangle + \langle R(X,Y)Y,X \rangle))dt. \tag{5.11}$$

From p. 91 [GKM] $\langle R(X,Y)Z,U \rangle = -\langle R(X,Y)U,Z \rangle$ and therefore $\langle R(X,Y)Z,Z \rangle = 0$. One has also $R(X,Y)Z = -R(Y,X)Z$ and therefore $R(X,X)Z = 0$.

Put $h(\theta,t) = \frac{1}{|X|^2}\langle Y,X\rangle$. Then $\tilde{Y} = Y - h.X$.

$$\langle R(\tilde{Y},X)X,\tilde{Y}\rangle = \langle R(Y,X)X,\tilde{Y}\rangle - \langle R(h.X,X)X,\tilde{Y}\rangle =$$

$$= \langle R(Y,X)X,Y\rangle - \langle R(Y,X)X,h.X\rangle = \langle R(Y,X)X,Y\rangle.$$

This shows that $\langle R(X,\tilde{Y})\tilde{Y},X\rangle = \langle R(X,Y)Y,X\rangle$. This changes (5.11) into

$$\partial_\theta^2 L(\theta) = \int_0^1 \frac{1}{|X|}\left(\langle\nabla_{\partial_t}\tilde{Y},\nabla_{\partial_t}\tilde{Y}\rangle - \langle R(X,\tilde{Y})\tilde{Y},X\rangle\right)dt + \int_0^1 \partial_t\langle\nabla_{\partial_\theta}Y, \frac{X}{|X|}\rangle dt =$$

$$= \int_0^1 \frac{1}{|X|}\left(\langle\nabla_{\partial_t}\tilde{Y},\nabla_{\partial_t}\tilde{Y}\rangle - \langle R(X,\tilde{Y})\tilde{Y},X\rangle\right)dt + \langle\tau(f)(\theta), \frac{X}{|X|}(\theta,1)\rangle \text{ (using (5.1))}$$

which is (5.5). $\square$

Step IV:

The integral $I = \int_0^1 \frac{1}{|X|}\left(\langle\nabla_{\partial_t}\tilde{Y},\nabla_{\partial_t}\tilde{Y}\rangle - \langle R(X,\tilde{Y})\tilde{Y},X\rangle\right)dt$

is non-negative.

Proof:

$I = \frac{1}{|X|}\int_0^1 \left(\langle\nabla_{\partial_t}\tilde{Y},\nabla_{\partial_t}\tilde{Y}\rangle - \langle R(X,\tilde{Y})\tilde{Y},X\rangle\right)dt$ since $|X|$ is independent of t.

Obviously if $\kappa_B \le 0$ the integral is non-negative.

Suppose $\kappa_B > 0$. As remarked above $\tilde{Y}(\theta,0) \equiv 0$. Consequently

$$|\tilde{Y}|^2(\theta,t) = \langle\tilde{Y},\tilde{Y}\rangle(\theta,t) - \langle\tilde{Y},\tilde{Y}\rangle(\theta,0) = \int_0^t \partial_t\langle\tilde{Y},\tilde{Y}\rangle dt = \int_0^t 2\langle\nabla_{\partial_t}\tilde{Y},\tilde{Y}\rangle dt \le$$

$$\le 2\int_0^t |\nabla_{\partial_t}\tilde{Y}||\tilde{Y}|dt \le 2\left(\int_0^1 |\nabla_{\partial_t}\tilde{Y}|^2 dt\right)^{\frac{1}{2}}\left(\int_0^1 |\tilde{Y}|^2 dt\right)^{\frac{1}{2}} .$$

Therefore,

$$\int_0^1 |\tilde{Y}|^2 dt \le 2\left(\int_0^1 |\nabla_{\partial_t}\tilde{Y}|^2 dt\right)^{\frac{1}{2}}\left(\int_0^1 |\tilde{Y}|^2 dt\right)^{\frac{1}{2}}$$

Or

$$\int_0^1 |\tilde{Y}|^2 dt \le 4\int_0^1 |\nabla_{\partial_t}\tilde{Y}|^2 dt \tag{5.12}$$

Since the radius r of B satisfies $r<(2\kappa_B^{\frac{1}{2}})^{-1}$ one has $|X| \le (2\kappa_B^{\frac{1}{2}})^{-1}$. Therefore

$$\langle R(X,\tilde{Y})\tilde{Y},X\rangle \le \kappa_B|X|^2|\tilde{Y}|^2 \le \frac{1}{4}|\tilde{Y}|^2.$$

Consequently

$$\int_0^1 \langle R(X,\tilde{Y})\tilde{Y},X\rangle dt \le \frac{1}{4}\int_0^1 |\tilde{Y}|^2 dt \le \int_0^1 |\nabla_{\partial_t}\tilde{Y}|^2 dt$$

using (5.12). This gives

$$\int_0^1 (|\nabla_{\partial_t}\tilde{Y}|^2 - \langle R(X,\tilde{Y})\tilde{Y},X\rangle)dt \ge 0. \qquad \square$$

By the corollary of Step II $\lim_{\theta\to 0^+} \partial_\theta L(\theta) = |\partial_\theta f(0)|$. From this one obtains

$$\partial_\theta L(\theta) = |\partial_\theta f(0)| + \int_0^\theta \partial_\theta^2 L(\theta')d\theta'.$$

From Steps III and IV there follows

$$\partial_\theta L(\theta) \geq |\partial_\theta f(0)| - \int_0^\theta |\tau(f)(\theta')|\, d\theta'.$$

Since f is a closed curve there must come a θ_1 where $\partial_\theta L(\theta_1) = 0$.

Then

$$|\partial_\theta f(0)| \leq \int_0^{\theta_1} |\tau(f)(\theta')|d\theta' \leq \int_0^{2\pi} |\tau(f)(\theta)|d\theta \leq (2\pi)^{\frac{1}{2}}(\int_0^{2\pi} |\tau(f)(\theta)|^2 d\theta)^{\frac{1}{2}}$$

and therefore

$$|\partial_\theta f(0)|^2 \leq 2\pi(\int_0^{2\pi} |\tau(f)(\theta)|^2 d\theta$$

By assumption the energy density attains its maximum at $\theta = 0$ so

$$E(f) \leq \pi|\partial_\theta f(0)|^2 \leq 2\pi^2 \int_0^{2\pi} |\tau(f)(\theta)|^2 d\theta. \quad \square$$

To conclude this section two examples of how one can combine the results of this and the preceding section.

Define $i:M \to \mathbb{R}_+$ by

$i(p) = \sup \{r:\exp_p T_p M \to M$ restricted to $B_r(0) \subset T_p M$ is a diffeomorphism onto $B_r(p)\}$.

Case 1:

There exists a compact $K \subset M$ such that $\kappa_{M\setminus K} \leq 0$ and $i|M\setminus K \geq 2r_o > 0$. Then any solution of the heat equation with initial curve f such that $m(f,\frac{1}{2\pi^2}) < r_o$ has bounded image.

Proof:

For any p such that $B_{3r_o}(p)$ lies wholly in $M\setminus K$, $B_{r_o}(p)$ has the property $P(\frac{1}{2\pi^2})$. For let g be a closed C^2 curve with image in $B_{r_o}(p)$, w.l.o.g. assume that the maximum of e(g) is attained at $\theta = 0$. $B_{2r_o}(g(0))$ is contained in $M\setminus K$ and so satisfies the hypotheses of Theorem 5A. Therefore Theorem 4A applies and the result follows.

Similarly:

Case 2:

There exists a compact $K \subset M$ such that $\kappa_{M\setminus K} > 0$ and $i|_{M\setminus K} \geq 2r_o \leq (2\kappa^{\frac{1}{2}}_{M\setminus K})^{-1}$. Then any solution of the heat equation with initial curve f such that $m(f,\frac{1}{2\pi^2}) < r_o$ has bounded image.

6. EXISTENCE OF SOLUTIONS OF THE HEAT EQUATION

This section will be devoted to the following existence problem: Given a closed C^1 curve on M does there exist a solution f_t of the heat equation defined for $t \in [0,t_1), t_1 \le \infty$, such that f_t and $\partial_\theta f_t$ are continuous at $t = 0$ and f_t coincides with the given curve at $t = 0$?

This problem will be treated under the assumption that the manifold M satisfies a condition (for example one from the previous sections) which will ensure that any solution of the kind sought will have its image contained in a fixed compact set. A solution of the heat equation will henceforth mean a solution of this problem for a fixed but arbitrary f_o.

The main ideas of this treatment are those of [ES]; therefore only an outline will be given with what is different here pointed out.

I. In [ES] it is shown how to replace the harmonic map equation $\tau(f) = 0$ and the heat equation $\partial_t f_t = \tau(f_t)$ (which in terms of local coordinates on M are local systems of equations) with global systems. This is done as follows: M can be smoothly and properly embedded in some Euclidean space $\mathbb{R}^q$ by a map $w:M \to \mathbb{R}^q$. Given such an embedding it is always possible to construct a smooth Riemannian metric on a tubular neighbourhood N of M so that N is Riemannian fibred. Let $\pi:N \to M$ be the projection map and π^c_{ab} its covariant differential. Then

(a) A map $f:S^1 \to M$ satisfies $\tau(f) = 0$ if and only if the composition $W = w \circ f$ satisfies

$$\partial_\theta{}^2 W^c = \pi^c_{ab}\ \partial_\theta W^a \partial_\theta W^b \ . \tag{6.1}$$

(b) A deformation $f_t: S^1 \to M$ $(t_0 < t < t_1)$ satisfies $\tau(f_t) = \partial_t f_t$ if and only if $W_t = w \circ f_t$ satisfies

$$\partial_\theta^2 W_t^c - \partial_t W_t^c = \pi_{ab}^c \partial_\theta W_t^a \partial_\theta W_t^b \tag{6.2}$$

Also, given a smooth $W_t: S^1 \to N$ satisfying (6.2) for $t_0 \leq t < t_1$, if W_{t_0} maps S^1 into M then so does every W_t for $t_0 \leq t < t_1$.

II. Derivative bounds

Lemma:

Any solution $f_t: S^1 \to M$ of the heat equation has energy density satisfying

$$\partial_\theta^2 e(f_t) - \partial_t e(f_t) = |\tau(f_t)|^2 \tag{6.3}$$

Proof:

$$\partial_\theta^2 e(f_t) = \partial_\theta^2 (\tfrac{1}{2} \langle \partial_\theta f_t, \partial_\theta f_t \rangle) =$$

$$= \partial_\theta \langle \nabla_{\partial_\theta} \partial_\theta f_t, \partial_\theta f_t \rangle = \langle \nabla_{\partial_\theta} \nabla_{\partial_\theta} \partial_\theta f_t, \partial_\theta f_t \rangle + \langle \nabla_{\partial_\theta} \partial_\theta f_t, \nabla_{\partial_\theta} \partial_\theta f_t \rangle$$

The heat equation can be written $\partial_t f_t = \nabla_{\partial_\theta} \partial_\theta f_t$ so

$$\partial_t e(f_t) = \langle \nabla_{\partial_t} \partial_\theta f_t, \partial_\theta f_t \rangle = \langle \nabla_{\partial_\theta} \partial_t f_t, \partial_\theta f_t \rangle = \langle \nabla_{\partial_\theta} \nabla_{\partial_\theta} \partial_\theta f_t, \partial_\theta f_t \rangle.$$

Therefore

$$\partial_\theta^2 e(f_t) - \partial_t e(f_t) = \langle \nabla_{\partial_\theta} \partial_\theta f_t, \nabla_{\partial_\theta} \partial_\theta f_t \rangle = |\tau(f_t)|^2. \quad \square$$

Put $H(\theta_1,\theta_2,t) = \frac{1}{2}(\pi t)^{-\frac{1}{2}} \exp\left[\frac{-(\theta_1-\theta_2)^2}{4t}\right]$ for $\theta_1,\theta_2 \in \mathbb{R}$, $t \in \mathbb{R}_+$.

H is a fundamental solution for the operator $L_\theta = \partial_\theta^2 - \partial_t$, satisfies $L_{\theta_1}H = L_{\theta_2}H = 0$, and the identity

$$u(\theta_1,t) = -\int_{t_0}^{t} d\tau \int_{\mathbb{R}} H(\theta_1,\theta_2,t-\tau) L_{\theta_2} u(\theta_2,\tau)\, d\theta_2$$

$$+ \int_{\mathbb{R}} H(\theta_1,\theta_2,t-t_0) u(\theta_2,t_0)\, d\theta_2 \quad t_0 < t < t_1 \tag{6.4}$$

holds for all u_t defined on S^1 which are of class C^2 in θ and C^1 in t for $t_0 \le t \le t_1$.

Suppose f_t is a solution of the heat equation defined for $0 \le t < t_1$. Since $H > 0$ there follows from (6.3) and (6.4)

$$e(f_t)(\theta_1) \le \int_{\mathbb{R}} H(\theta_1,\theta_2,t-t_0) e(f_{t_0})(\theta_2)\, d\theta_2 \text{ for } 0 < t_0 < t < t_1. \tag{6.5}$$

For $t > 1$, putting $t-1$ for t_0 in (6.5) there follows

$$e(f_t)(\theta_1) \le \int_{\mathbb{R}} H(\theta_1,\theta_2,1) e(f_{t-1})(\theta_2)\, d\theta_2$$

and therefore

$$e(f_t) \le \text{const} \int_0^{2\pi} e(f_{t-1})(\theta)\, d\theta.$$

Any smaller value can be put in for t-1 on the right for example zero since $e(f_t)$ is assumed to be continuous at $t = 0$.

For $0 < t \leq 1$ put $t_o = 0$ in (6.5) to obtain

$$e(f_t)(\theta_1) \leq \int_{\mathbb{R}} H(\theta_1,\theta_2,t)e(f_o)(\theta_2)d\theta_2 \tag{6.6}$$

Put $\bar{e}(f_o) = \sup_{\theta \in S^1} e(f_o)(\theta)$ then (6.6) shows that

$$e(f_t)(\theta) \leq \text{const. } \bar{e}(f_o).$$

To summarize:

<u>Theorem 6A</u>

Let f_t be a solution of the heat equation for $0 \leq t < t_1$. Then

$$e(f_t) \leq \text{const.} \int_0^{2\pi} e(f_o)(\theta)d\theta \quad \text{for } 1 \leq t < t_1$$

$$e(f_t) \leq \text{const. } \sup_{\theta \in S^1} e(f_o)(\theta) \quad \text{for } 0 \leq t \leq 1$$

with the constants not depending on f_t.

The difference between this and the derivation of bounds for the first order space derivatives in [ES] is that the curvature terms in the identity (6) §8A in [ES] do not appear in the corresponding identity here (i.e. (6.3)), because the domain is 1-dimensional. It is therefore not necessary to impose curvature restrictions on the target manifold M.

For a solution f_t of the heat equation there is the following theorem, proved in [ES], for the second derivative with respect to θ of the W_t in (6.2).

Theorem 6B

Given ε there is a constant C independent of t such that $|\partial_\theta^2 W_t^c| < C$ for $t > \varepsilon$. The constant C depends on f_0.

This is proved by using the formula

$$W_t^c(\theta) = -\int_0^t d\tau \int_{\mathbb{R}} H(\theta,\theta',t-\tau)F^c(\theta',\tau)d\theta' + W_0^c(\theta,t)$$

where

$$W_0^c(\theta,t) = \int_{\mathbb{R}} H(\theta,\theta',t)W^c(\theta',0)d\theta'$$

and the F^c the functions on the right of (6.2) and the properties of the fundamental solution H. The only remark that should be made is that here it is assumed that the manifold satisfies conditions which will ensure that the image of any solution will be contained in a fixed compact set and therefore the embedding conditions in [ES] are not necessary since the inequalities (12), §8D in [ES] are automatically satisfied on a compact set.

III The following two theorems are proved in §10 [ES].

Theorem 6C

If f_t and f'_t are two solutions of the heat equation with $f_0 = f'_0$ then they coincide for all relevant $t > 0$.

<u>Theorem 6D</u>

Let M' be a compact subset of M. Then for any closed C^1 curve $f_0:S^1 \to M$ such that $f_0(S^1)$ lies in M' there is a positive constant t_1 depending only on M' and the magnitude of the energy density $e(f_0)$ such that there exists a solution f_t for that f_0 for $0 \leq t \leq t_1$.

From these two theorems one deduces the following which corresponds to theorem 10C in [ES].

<u>Theorem 6E</u>

There is a unique solution f_t of the heat equation defined for all $t \geq 0$.

<u>Proof</u>

Such a solution exists for small t by Theorem 6D and is unique by Theorem 6C. Let t_1 be the largest number such that a solution of the kind sought exists for $0 \leq t < t_1$ and suppose that t_1 is finite. By assumption the manifold M satisfies conditions which ensure that the images $f_t(S^1)$ $(0 \leq t < t_1)$ all lie in a compact subset of M. Theorem 6A shows that the energy density remains bounded and therefore by Theorems 6C and 6D there is a fixed positive number ε such that any f_t can be continued as a solution into the interval $(t, t+\varepsilon)$. This contradicts the finiteness of t_1.

7. SUBCONVERGENCE OF SOLUTIONS

In this section let f_o be a fixed closed C^1 curve $S^1 \to M$ and assume that M satisfies conditions which will ensure that any solution of the heat equation f_t which is continuous along with $\partial_\theta f_t$ at $t = 0$ and which coincides with the given f_o at $t = 0$ will have its image contained in a fixed compact set. Then by the preceding section a unique solution f_t exists for all $t \in [0,\infty)$. In this section a proof of the following:

Theorem 7A

There is a sequence $t_1, t_2, t_3, \ldots$ with $t_k \to \infty$ such that the curves $f_k = f_{t_k}$ converge uniformly to a closed geodesic f.

Lemma (See [H])

For $k(f_t) = \frac{1}{2}\langle \partial_t f_t, \partial_t f_t \rangle$ one has the following identity

$$\partial_t k(f_t) = \partial_\theta^2 k(f_t) - |\nabla_{\partial_\theta} \partial_t f|^2 + \langle R(\partial_\theta f_t, \partial_t f_t)\partial_\theta f_t, \partial_t f_t \rangle \tag{7.1}$$

Proof

$$\partial_\theta^2 [\tfrac{1}{2}\langle \partial_t f_t, \partial_t f_t \rangle] = \partial_\theta \langle \nabla_{\partial_\theta} \partial_t f_t, \partial_t f_t \rangle =$$

$$= \langle \nabla_{\partial_\theta} \nabla_{\partial_\theta} \partial_t f_t, \partial_t f_t \rangle + \langle \nabla_{\partial_\theta} \partial_t f_t, \nabla_{\partial_\theta} \partial_t f_t \rangle =$$

$$= \langle \nabla_{\partial_\theta} \nabla_{\partial_t} \partial_\theta f_t, \partial_t f_t \rangle + |\nabla_{\partial_\theta} \partial_t f_t|^2.$$

Also,

$$\partial_t[\tfrac{1}{2}\langle\partial_t f_t,\partial_t f_t\rangle] = \langle\nabla_{\partial_t}\partial_t f_t,\partial_t f_t\rangle =$$

$$= \langle\nabla_{\partial_t}\nabla_{\partial_\theta}\partial_\theta f_t,\partial_t f_t\rangle$$

Combining these it is easily seen that

$$\langle\nabla_{\partial_t}\nabla_{\partial_\theta}\partial_\theta f_t,\partial_t f_t\rangle = \partial_\theta^2[\tfrac{1}{2}\langle\partial_t f_t,\partial_t f_t\rangle] - |\nabla_{\partial_\theta}\partial_t f_t|^2 + \langle\nabla_{\partial_t}\nabla_{\partial_\theta}\partial_\theta f_t - \nabla_{\partial_\theta}\nabla_{\partial_t}\partial_\theta f_t,\partial_t f_t\rangle$$

which is (7.1). □

<u>Lemma</u>

$\partial_t E(t) \to 0$ as $t \to \infty$. (As before $E(t) = E(f_t)$).

<u>Proof</u>

Put $K(f_t) = \int_0^{2\pi} k(f_t)(\theta)d\theta$ with $k(f_t)$ as in preceding lemma. $\partial_t E(t) = -2K(f_t)$ (Corollary of Lemma 3A) and

$$\partial_t^2 E(t) = -2\int_0^{2\pi}\partial_t k(f_t)(\theta)d\theta.$$

The last term in (7.1) is uniformly bounded for t greater than any given positive number by Theorems 6A and 6B. By integrating (7.1) over S^1 it therefore follows that there exists a constant C such that $\partial_t^2 E(t) \geq C$. $\partial_t E(t)$ cannot be bounded away from zero because it is integrable so if $C \geq 0$ then obviously $\partial_t E(t) \to 0$. Suppose $C < 0$. If for some $\varepsilon > 0$ there exist arbitrarily large t_0 such that $\partial_t E(t_0) = -\varepsilon$ then

$$\int_{t_o+\varepsilon/C}^{t_o} \partial_t E(t)dt \le \int_{t_o+\varepsilon/C}^{t_o} (-\varepsilon - C(t_o - t)) = \varepsilon^2/2C$$

but this contradicts the integrability of $\partial_t E(t)$. □

Remark

This lemma will be used in the proof of Theorem 7A. In [ES] the corresponding result (Corollary §6(C)) is proved with the assumption that the target manifold has non-positive sectional curvature. This assumption is not necessary here.

In the following proof it will be convenient to use the function G defined by

$$G(\theta,\theta') = \int_0^{2\pi} \chi_{(\theta,2\pi)}(r)\chi_{(0,r)}(\theta')dr$$

where $\chi_{(a,b)}$ is the characteristic function of the interval (a,b). The formula

$$h(\theta) = h(0) - \partial_\theta h(0)(2\pi-\theta) - \int_0^{2\pi} G(\theta,\theta')\partial_\theta^2 h(\theta')d\theta'$$

holds for all C^2 functions $h:S^1 \to \mathbb{R}$ and

$$\partial_\theta^2 \int_0^{2\pi} G(\theta,\theta')h(\theta')d\theta' = -h(\theta) \tag{7.2}$$

holds for all continuous functions $h:S^1 \to \mathbb{R}$.

Proof of Theorem 7A

Let $W_t = w\circ f_t$ be the solution of (6.2) which corresponds to f_t. The mappings W_t and $\partial_\theta W_t$ form bounded equicontinuous families (Theorems 6A and 6B). Therefore there exists a sequence $t_1, t_2, t_3, \ldots$ with $t_k \to \infty$ such that the mappings $W_k = W_{t_k}$ converge uniformly along with $\partial_\theta W_k$ to a continuously differentiable mapping W. The W_k can be represented by the formula

$$W_k^c(\theta) = W_k^c(0) - \partial_\theta W_k^c(0)(2\pi-\theta) - \int_0^{2\pi} G(\theta,\theta')\partial_\theta^2 W_k^c(\theta')d\theta'$$

or

$$W_k^c(\theta) = W_k^c(0) - \partial_\theta W_k^c(0)(2\pi-\theta) - \int_0^{2\pi} G(\theta,\theta')(F_k^c(\theta') + \partial_t W_k^c(\theta'))d\theta' \tag{7.3}$$

where $F_k^c = \pi_{ab}^c\, \partial_\theta W_k^a\, \partial_\theta W_k^b$. By the preceding lemma the $\partial_t W_k^c(\theta)$ converge in the mean to zero as $k \to \infty$. Therefore, since G is bounded

$$\lim_{k\to\infty} \int_0^{2\pi} G(\theta,\theta')\partial_t W_k^c(\theta')d\theta' = 0.$$

Passing to the limit in (7.3) there results for the mapping W

$$W^c(\theta) = W^c(0) - \partial_\theta W^c(0)(2\pi-\theta) - \int_0^{2\pi} G(\theta,\theta')F^c(\theta')d\theta'$$

where $F^c(\theta') = \lim_{k\to\infty} F_k^c(\theta') = \pi_{ab}^c(W)\partial_\theta W^a \partial_\theta W^b$. Referring to (7.2) it is seen that W satisfies (6.1) which means that it corresponds to a closed geodesic. □

PART II

1. CONTENTS

The material of this part is arranged in sections as follows: after some preliminary material in Section 2, Section 3 is about the decomposition of Lorentz vector spaces associated to timelike vectors and about Riemannian metrics associated to a time-orientable Lorentz manifold. Section 4 is about the evolution of the energy and length of solution curves while Section 5 is devoted to examples. Sections 6, 7 and 8 contain the discussion about the existence and uniqueness of solutions and Section 9 is about the subconvergence of timelike solutions.

2. PRELIMINARIES

Definition

A Lorentz manifold (M,g) is a smooth manifold M of dimension $m \geq 2$ with a symmetric nondegenerate $(0,2)$ tensor field g of constant index $\nu = 1$.

The index ν of a symmetric bilinear form b on a vector space V is the largest integer that is the dimension of a subspace $W \subset V$ on which $b|W$ is negative definite.

$g(u,v)$ will usually be written $\langle u,v \rangle$.

As for Riemannian manifolds one has the Levi-Civita connection associated to Lorentz manifolds and the same notation and properties as specified in Section 2 of Part I will be used. In particular the geodesic equation is written the same way

$$\nabla_{\partial_\theta} \partial_\theta f = 0 \tag{2.1}$$

Definition

A pregeodesic is a curve that has a reparametrization as a geodesic.

The following result will be used ([O'N] pp. 69, 95-96).

Lemma 2A:

A curve f is a pregeodesic if and only if the tension field $\nabla_{\partial_\theta} \partial_\theta f$ and the tangent field $\partial_\theta f$ are everywhere collinear.

<u>Definitions</u>

A tangent vector v to a Lorentz manifold $(M, \langle\ ,\ \rangle)$ is

spacelike if $\langle v,v\rangle > 0$ or $v = 0$

lightlike or null if $\langle v,v\rangle = 0$ and $v \neq 0$

timelike if $\langle v,v\rangle < 0$.

The light cone at p is the set $\{v \in T_pM : v \text{ lightlike}\}$
A curve α on M is spacelike if all of its velocity vectors are spacelike; similarly for timelike and lightlike.

On a time-orientable Lorentz manifold M [O'N pp.144-145] there exists a smooth timelike vector field γ.

<u>Definitions</u>

Let $f : S^1 \to M$ be a curve on the Lorentz manifold $(M, \langle\ ,\ \rangle)$.
The energy density $e(f)(\theta) = \frac{1}{2}\langle\partial_\theta f(\theta), \partial_\theta f(\theta)\rangle$, the energy of f
$E(f) = \int_{S^1} e(f)(\theta)d\theta$. When f is timelike define $\ell(f)(\theta) = (-\langle\partial_\theta f(\theta), \partial_\theta f(\theta)\rangle)^{\frac{1}{2}}$ and $L(f) = \int_{S^1} \ell(f)(\theta)d\theta$.

Unlike the Riemannian situation the quantities $e(f)$ and $E(f)$ can now take on all real values. The quantity $L(f)$ is unchanged by monotone reparametrizations [O'N p. 132].

3. RIEMANNIAN METRICS ASSOCIATED TO A TIME-ORIENTABLE LORENTZ MANIFOLD

I. Lemma 3A [O'N p.141]: Let γ be a timelike vector in a Lorentz vector space $(V, \langle\ ,\ \rangle)$ (i.e. V is a vector space on which is defined a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ of index 1). Then the subspace $\gamma^{\perp}$ (i.e. $\{v \in V : \langle v,\gamma\rangle = 0\}$) is spacelike and V is the direct sum $\mathbf{R}\cdot\gamma + \gamma^{\perp}$. □

Let $(V, \langle\ ,\ \rangle)$ be a Lorentz vector space and γ a timelike vector in V. For any vector X in V, its decomposition relative to γ can be written

$$X = \frac{\langle X,\gamma\rangle}{\langle\gamma,\gamma\rangle}\gamma + \left[X - \frac{\langle X,\gamma\rangle}{\langle\gamma,\gamma\rangle}\gamma\right] \tag{3.1}$$

One has then

$$\langle X,X\rangle = \frac{\langle X,\gamma\rangle^2}{\langle\gamma,\gamma\rangle} + \left\langle X - \frac{\langle X,\gamma\rangle}{\langle\gamma,\gamma\rangle}\gamma,\ X - \frac{\langle X,\gamma\rangle}{\langle\gamma,\gamma\rangle}\gamma\right\rangle \tag{3.2}$$

where the first term on the right is negative and the second term positive.

II. Let $(M, \langle\ ,\ \rangle)$ be a time-orientable Lorentz manifold. Then there exists a timelike vector field γ on M. One can assume that $\langle\gamma,\gamma\rangle \equiv -1$. Then one has (see [A])

Lemma 3B: $\langle\ ,\ \rangle_\gamma$ defined by

$$\langle X,Y\rangle_\gamma = \langle X,Y\rangle + 2\langle X,\gamma\rangle\langle Y,\gamma\rangle$$

defines a Riemannian metric on M. □

It is easily seen that

$$\langle X,Y\rangle = \langle X,Y\rangle_\gamma - 2\langle X,\gamma\rangle_\gamma \langle Y,\gamma\rangle_\gamma$$

and that for a timelike vector X

$$||X||_\gamma = (\langle X,X\rangle_\gamma)^{\frac{1}{2}} \leq (2\langle X,\gamma\rangle^2)^{\frac{1}{2}} = \sqrt{2}\,|\langle X,\gamma\rangle|$$

and (3.3)

$$(-\langle X,X\rangle)^{\frac{1}{2}} \leq \sqrt{2}\,|\langle X,\gamma\rangle|$$

Let ∇^γ denote the Levi-Civita connection associated to $\langle\ ,\ \rangle_\gamma$.

Proposition 3C

$\nabla^\gamma = \nabla$ if and only if γ is parallel i.e. the map $X \to \nabla_X\gamma$ is identically zero.

Proof

Since $\langle\gamma,\gamma\rangle \equiv -1$ one has

$$\langle\nabla_X\gamma,\gamma\rangle = 0 \tag{3.4}$$

for every tangent vector X.

Let X, Y, Z denote vector fields on M. It is easy to show that

$$\langle\nabla^\gamma_X Y,Z\rangle_\gamma = \langle\nabla_X Y,Z\rangle_\gamma + [\langle Z,\gamma\rangle(\langle Y,\nabla_X\gamma\rangle + \langle X,\nabla_Y\gamma\rangle) +$$

$$+ \langle Y,\gamma\rangle(\langle Z,\nabla_X\gamma\rangle - \langle X,\nabla_Z\gamma\rangle) + \langle X,\gamma\rangle(\langle Z,\nabla_Y\gamma\rangle - \langle Y,\nabla_Z\gamma\rangle)] \tag{3.5}$$

Clearly $\nabla = \nabla^\gamma$ if and only if the expression in the bracket in (3.5) vanishes for all X, Y, Z. Therefore it is immediate that γ parallel implies $\nabla = \nabla^\gamma$.

Conversely, suppose that the expression in the bracket vanishes for all X, Y, Z. (This is a pointwise condition, the expression is linear in X, Y and Z, it therefore suffices to consider tangent vectors at each point).

Choosing $Z = \gamma$ and $Y, X \perp \gamma$ in (3.5) one sees that

$$\langle Y, \nabla_X \gamma \rangle = -\langle X, \nabla_Y \gamma \rangle \text{ for all } X, Y \perp \gamma.$$

Similarly choosing $Y = \gamma$ and $X, Z \perp \gamma$ in (3.5) one obtains

$$\langle Z, \nabla_X \gamma \rangle = \langle X, \nabla_Z \gamma \rangle \text{ for all } X, Z \perp \gamma.$$

The conclusion is that

$$\langle X, \nabla_Y \gamma \rangle = 0 \text{ for all } X, Y \perp \gamma. \tag{3.6}$$

Now, choosing $Z \perp \gamma$, $X = Y = \gamma$ in (3.5) there follows

$$\langle Z, \nabla_\gamma \gamma \rangle = 0 \text{ for all } Z \perp \gamma. \tag{3.7}$$

From (3.4), (3.6) and (3.7) there follows that $\nabla_X \gamma \equiv 0$. $\Box$

4. THE HEAT EQUATION

The heat equation for closed curves on Lorentz manifolds is defined in the same way as for Riemannian manifolds i.e.

$$\nabla_{\partial_\theta}\partial_\theta f_s = \partial_s f_s \tag{4.1}$$

This equation and the geodesic equation (2.1) have the same form as the corresponding Riemannian equations and will therefore have the same smoothness properties as quoted at the end of Section 3 of Part I.

I. Proposition 4A:

Let $f_s : S^1 \to M$ be a solution of the heat equation for s in some interval $I \subset \mathbb{R}$. If there exists an $s_o \in I$ s.t. $f_{s_o} : S^1 \to M$ is a timelike curve then the f_s will also be timelike for all subsequent $s \in I$.

Proof

Suppose that $f_s : S^1 \to M$ is timelike and that

$$\max\{\langle\partial_\theta f_s, \partial_\theta f_s\rangle : \theta \in S^1\} < 0 \text{ occurs at } \theta_o.$$

Then

$$\partial_\theta\langle\partial_\theta f_s, \partial_\theta f_s\rangle|_{\theta=\theta_o} = 2\langle\nabla_{\partial_\theta}\partial_\theta f_s, \partial_\theta f_s\rangle|_{\theta=\theta_o} = 0 \tag{4.2}$$

and

$$\partial_\theta^2\langle\partial_\theta f_s, \partial_\theta f_s\rangle|_{\theta=\theta_o} = 2[\langle\nabla_{\partial_\theta}\nabla_{\partial_\theta}\partial_\theta f_s, \partial_\theta f_s\rangle + \langle\nabla_{\partial_\theta}\partial_\theta f_s, \nabla_{\partial_\theta}\partial_\theta f_s\rangle]_{\theta=\theta_o} \leq 0 \tag{4.3}$$

Further

$$\partial_s\langle\partial_\theta f_s,\partial_\theta f_s\rangle|_{\theta=\theta_0} = 2\langle\nabla_{\partial_s}\partial_\theta f,\partial_\theta f_s\rangle|_{\theta=\theta_0} =$$

$$= 2\langle\nabla_{\partial_\theta}\partial_s f_s,\partial_\theta f_s\rangle|_{\theta=\theta_0} = 2\ \langle\nabla_{\partial_\theta}\nabla_{\partial_\theta}\partial_\theta f_s,\partial_\theta f_s\rangle|_{\theta=\theta_0}\ . \tag{4.4}$$

Now, since $\nabla_{\partial_\theta}\partial_\theta f_s|_{\theta=\theta_0} \in \partial_\theta f_s(\theta_0)^\perp$ by (4.2) one has $\langle\nabla_{\partial_\theta}\partial_\theta f_s,\nabla_{\partial_\theta}\partial_\theta f_s\rangle|_{\theta=\theta_0} \geq 0$ by Lemma 3A, and therefore by (4.3) that $\langle\nabla_{\partial_\theta}\nabla_{\partial_\theta}\partial_\theta f_s,\partial_\theta f_s\rangle|_{\theta=\theta_0} \leq 0$. By (4.4) this implies that $\partial_s\langle\partial_\theta f_s,\partial_\theta f_s\rangle|_{\theta=\theta_0} \leq 0$. $\square$

Remarks

The above Proposition shows that deformations by heat flow of timelike curves defines a t-homotopy i.e. a homotopy through timelike curves.

There is no corresponding result for the minima of the energy density of solutions (see Example IA in the next section).

The property of being spacelike is not preserved by the heat flow (Example IIIB, next section).

II. Putting $E(s) = E(f_s)$ one obtains in the same way as in the Riemannian case (Part I, Section 3) the formula

$$\partial_s E(s) = -\int_0^{2\pi} \langle\nabla_{\partial_\theta}\partial_\theta f_s,\nabla_{\partial_\theta}\partial_\theta f_s\rangle d\theta \tag{4.5}$$

Because of the nondefinite nature of $\langle\ ,\ \rangle$ this does not give any result analogous to the corollary of Section 3, Part I.

For the remainder of this section let f_s denote a timelike solution of the heat equation. Decomposing $\nabla_{\partial_\theta}\partial_\theta f_s$ into factors parallel and orthogonal to $\partial_\theta f_s$ (Section 3I) one can write (4.5) as

$$\partial_s E(s) = \int_0^{2\pi} - \frac{\langle \nabla_{\partial_\theta}\partial_\theta f_s, \partial_\theta f_s\rangle^2}{\langle \partial_\theta f_s, \partial_\theta f_s\rangle} d\theta \; +$$

$$+ \int_0^{2\pi} - \langle \nabla_{\partial_\theta}\partial_\theta f_s - \frac{\langle \nabla_{\partial_\theta}\partial_\theta f_s, \partial_\theta f_s\rangle}{\langle \partial_\theta f_s, \partial_\theta f_s\rangle} \partial_\theta f_s , \nabla_{\partial_\theta}\partial_\theta f_s - \frac{\langle \nabla_{\partial_\theta}\partial_\theta f_s, \partial_\theta f_s\rangle}{\langle \partial_\theta f_s, \partial_\theta f_s\rangle} \partial_\theta f_s \rangle d\theta \tag{4.6}$$

(these integrals are well defined since $\langle \partial_\theta f_s, \partial_\theta f_s\rangle$ is bounded away from zero). The first integral on the right of (4.6) is always non-negative and the second always non-positive. Using (4.6) one can prove

<u>Proposition 4B</u>

Suppose that each of the curves $f_s : S^1 \to M$ is of constant energy density. Then $\partial_s E(s) \leq 0$ for all s, with $\partial_s E(s_o) = 0$ for some s_o only if f_{s_o} is a closed geodesic.

<u>Proof</u>

By hypothesis $\langle \partial_\theta f_s, \partial_\theta f_s\rangle$ depends only on s. Therefore

$$\partial_\theta \tfrac{1}{2} \langle \partial_\theta f_s, \partial_\theta f_s\rangle = \langle \nabla_{\partial_\theta}\partial_\theta f_s, \partial_\theta f_s\rangle \equiv 0 \tag{4.7}$$

(4.7) shows that the first integral in (4.6) vanishes for all s. Therefore $\partial_s E(s) \leq 0$ for all s. (4.7) also shows that $\nabla_{\partial_\theta}\partial_\theta f_s$ is everywhere spacelike. Therefore one can only have $\partial_s E(s_o) = 0$ if $\nabla_{\partial_\theta}\partial_\theta f_{s_o} = 0$ for all $\theta \in S^1$ in other words if f_{s_o} is a geodesic. □

Put

$$\varepsilon_{max}(s) = \min_{\theta\in S^1} \{\langle\partial_\theta f_s, \partial_\theta f_s\rangle\} \text{ and } \varepsilon_{min}(s) = \sup_{\theta\in S^1} \{\langle\partial_\theta f_s, \partial_\theta f_s\rangle\}$$

For the first integral in (4.6) one has the following estimate

<u>Proposition</u>

$$\frac{2}{\pi}\left((-\varepsilon_{max}(s))^{\frac12} - (-\varepsilon_{min}(s))^{\frac12}\right)^2 \leq \int_0^{2\pi} - \frac{\langle\nabla_{\partial_\theta}\partial_\theta f_s, \partial_\theta f_s\rangle^2}{\langle\partial_\theta f_s, \partial_\theta f_s\rangle}\, d\theta$$

<u>Proof</u>

$$(2((-\varepsilon_{max}(s))^{\frac12} - (-\varepsilon_{min}(s))^{\frac12}))^2 \leq \left(\int_0^{2\pi} |\partial_\theta(-\langle\partial_\theta f_s, \partial_\theta f_s\rangle)^{\frac12}|\right)^2 d\theta =$$

$$= \left(\int_0^{2\pi} |(-\langle\partial_\theta f_s, \partial_\theta f_s\rangle)^{-\frac12}\langle\nabla_{\partial_\theta}\partial_\theta f_s, \partial_\theta f_s\rangle|\right)^2 \leq 2\pi \int_0^{2\pi} - \frac{\langle\nabla_{\partial_\theta}\partial_\theta f_s, \partial_\theta f_s\rangle^2}{\langle\partial_\theta f_s, \partial_\theta f_s\rangle}\, d\theta \quad \square$$

III. Put $L(s) = L(f_s) = \int_0^{2\pi} (-\langle\partial_\theta f_s, \partial_\theta f_s\rangle)^{\frac12}\, d\theta$.

<u>Proposition 4C</u>

$$\partial_s L(s) = \int_0^{2\pi} (-\langle\partial_\theta f_s, \partial_\theta f_s\rangle)^{-\frac12}\left[\langle\nabla_{\partial_\theta}\partial_\theta f_s, \nabla_{\partial_\theta}\partial_\theta f_s\rangle - \frac{\langle\nabla_{\partial_\theta}\partial_\theta f_s, \partial_\theta f_s\rangle^2}{\langle\partial_\theta f_s, \partial_\theta f_s\rangle}\right] d\theta$$

(4.8)

Proof

$$\partial_s L(s) = \int_0^{2\pi} \partial_s(-\langle\partial_\theta f_s,\partial_\theta f_s\rangle)^{\frac12} d\theta = -\int_0^{2\pi}(-\langle\partial_\theta f_s,\partial_\theta f_s\rangle)^{-\frac12}\langle\nabla_{\partial_s}\partial_\theta f_s,\partial_\theta f_s\rangle d\theta =$$

$$= -\int_0^{2\pi}(-\langle\partial_\theta f_s,\partial_\theta f_s\rangle)^{-\frac12}\langle\nabla_{\partial_\theta}\nabla_{\partial_\theta}\partial_\theta f_s,\partial_\theta f_s\rangle d\theta \qquad (4.9)$$

Also,

$$\partial_\theta[((-\langle\partial_\theta f_s,\partial_\theta f_s\rangle)^{-\frac12})\langle\nabla_{\partial_\theta}\partial_\theta f_s,\partial_\theta f_s\rangle] = \frac{\langle\nabla_{\partial_\theta}\nabla_{\partial_\theta}\partial_\theta f_s,\partial_\theta f_s\rangle}{(-\langle\partial_\theta f_s,\partial_\theta f_s\rangle)^{\frac12}} +$$

$$+ \frac{\langle\nabla_{\partial_\theta}\partial_\theta f_s,\nabla_{\partial_\theta}\partial_\theta f_s\rangle}{(-\langle\partial_\theta f_s,\partial_\theta f_s\rangle)^{\frac12}} + \frac{\langle\nabla_{\partial_\theta}\partial_\theta f_s,\partial_\theta f_s\rangle^2}{(-\langle\partial_\theta f_s,\partial_\theta f_s\rangle)^{3/2}} .$$

Substituting this into (4.9) gives (4.8). □

Corollary 4D

$\partial_s L(s) \geq 0$ for all s and $\partial_s L(s_0) = 0$ only if f_{s_0} is a pregeodesic.

Proof

Write $\nabla_{\partial_\theta}\partial_\theta f_s = \tau_s^{\parallel}(\theta) + \tau_s^{\perp}(\theta)$ where $\tau_s^{\parallel}(\theta)$ (resp. $\tau_s^{\perp}(\theta)$) is parallel (resp. orthogonal) to $\partial_\theta f_s$. $\tau^{\perp}$ is then of course spacelike. The expression inside the bracket in the integral on the right hand side of (4.8) equals $\langle\tau^{\perp},\tau^{\perp}\rangle \geq 0$. $\langle\tau_{s_0}^{\perp},\tau_{s_0}^{\perp}\rangle$ equals zero only when $\tau_{s_0}^{\perp} = 0$ which implies by Lemma 2A that f_{s_0} is a pregeodesic. □

5. EXAMPLES

In this section a few examples are given of solutions of heat equations and their properties to illustrate some kinds of behaviour that can occur and some differences and similarities between the Lorentz and Riemannian situations.

I. Let M be the manifold $\mathbb{R} \times S^1$ parametrized by $r \in \mathbb{R}$ and the central angle $\phi \in S^1$. Let $h : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ be a smooth function. Define a metric tensor g on M by $ds^2 = dr^2 + h(r)d\phi^2$. When $h > 0$ this gives a Riemannian metric on M, when $h < 0$ a Lorentz metric.

One has

$$g_{11} = \langle \partial_\phi, \partial_\phi \rangle = h,\ g_{12} = g_{21} = 0,\ g_{22} = \langle \partial_r, \partial_r \rangle = 1$$

By using the formulas given in [O'N p. 80] one can calculate the Christoffel symbols of g. They are

$$\Gamma^2_{11} = -\frac{1}{2}h' \qquad \Gamma^1_{12} = \frac{h'}{2h}$$

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = \Gamma^2_{22} = 0.$$

The heat equation can therefore be written

$$\left.\begin{aligned} \partial_\theta{}^2\phi + \frac{h'}{h}\ \partial_\theta r\ \partial_\theta\phi &= \partial_s\phi \\ \partial_\theta{}^2 r - \frac{1}{2}h'(\partial_\theta\phi)^2 &= \partial_s r \end{aligned}\right\} \tag{5.1}$$

Suppose that the function ρ_{r_0} satisfies

$$\left.\begin{aligned} \partial_s \rho_{r_o}(s) &= -\frac{h'}{2}(\rho_{r_o}(s)) \\ \rho_{r_o}(0) &= r_o \end{aligned}\right\} \qquad (5.2)$$

for s in some maximal interval I containing 0.

Define $f : S^1 \times I \to M$ by $(\theta,s) \to (\rho_{r_o}(s),\theta)$. Then f is a solution of the heat equation (5.1). As is easy to see from (5.1) and (5.2) the map $S^1 \times (-I) \to M$ defined by $(\theta,s) \to f(\theta,-s)$ is a solution of the heat equation for the metric $ds'^2 = dr^2 - h(r)d\phi^2$. [Remark: this last statement is not true for general solutions f; as is seen from (5.1) it depends on the property that $\partial_\theta r \equiv 0$].

The following specific examples will illustrate some properties of solutions of the heat equation for Lorentz metrics and differences from the Riemannian situation.

A. Take $h = \pm e^r$. The differential equations

$$\left.\begin{aligned} \partial_s \rho^{\pm}_{r_o} &= \mp \frac{1}{2} e^{\rho_{r_o}} \\ \rho^{\pm}_{r_o}(0) &= r_o \end{aligned}\right\}$$

have the solutions

$$\rho^{\pm}_{r_o}(s) = -\ln(e^{-r_o} \pm \frac{1}{2}s)$$

where $\rho^{-}_{r_o}$ (resp. $\rho^{+}_{r_o}$) is defined on $I^- = (-\infty, 2e^{-r_o})$ (resp. $I^+ = (-2e^{-r_o},\infty)$). $f^{\pm}_{r_o} : S^1 \times I^{\pm} \to M, (\theta,s) \to (-\ln(e^{-r_o} \pm \frac{1}{2}s),\theta)$ is then a solution of the heat equation for the metric $ds^2_{\pm} = dr^2 \pm e^r d\phi^2$. For the energy density one has

$$e(f^{\pm}_{r_o})(\theta,s) = \frac{1}{2}\langle\partial_\theta f^{\pm}_{r_o},\partial_\theta f^{\pm}_{r_o}\rangle(\theta,s) = \pm\,\frac{1}{2}(e^{-r_o} \pm \frac{1}{2}s)^{-1}$$

This example shows that solutions of the heat equation for Lorentz metrics need not exist for all positive values of the deformation parameter s. Also, in this example one has for $E(s) = E(f^{-}_{r_o s})$

$$\partial_s E(s) = -\frac{1}{4}(e^{-r_o} - \frac{1}{2}s)^{-2} < 0 \text{ and } \partial_s^2 E(s) = -\frac{1}{4}(e^{-r_o} - \frac{1}{2}s)^{-3} < 0.$$

B. Let M be the manifold $\mathbb{R}_+ \times S^1$ with the Lorentz metric $ds^2 = dr^2 - r^a d\phi^2, a \in \mathbb{R} \setminus \{0\}$. The equation

$$\left.\begin{aligned} \partial_s\rho_{r_o} &= \frac{a}{2}\,\rho_{r_o}^{a-1} \\ \rho_{r_o}(0) &= r_o \end{aligned}\right\}$$

has the solution

$$\rho_{r_o}(s) = \begin{cases} (r_o^{2-a} + \frac{1}{2}a(2-a)s)^{\frac{1}{2-a}} & a \neq 2 \\ r_o e^s & a = 2 \end{cases}$$

For $a < 0$ the solution exists for $s \in I^a = (-\infty, -\frac{2r_o^{2-a}}{a(2-a)})$ and $\rho_{r_o} \to 0$ as $s \to \frac{-2r_o^{2-a}}{a(2-a)}$ and $\lim\limits_{s\to-\infty} \rho_{r_o}(s) = \infty$.

(For $a = 0$ the map $(\theta,s) \to (r_o,\theta)$ is a solution of the heat equation). For $0 < a < 2$ the solution exists for $s \in I^a = (-\frac{2r_o^{2-a}}{a(2-a)},\infty)$ and $\rho_{r_s}(s) \to \infty$ as $s \to \infty$, $\rho_{r_o}(s) \to 0$ as $s \to \frac{-2r_o^{2-a}}{a(2-a)}$.

$$e(f^{\pm}_{r_o})(\theta,s) = \tfrac{1}{2}\langle\partial_\theta f^{\pm}_{r_o},\partial_\theta f^{\pm}_{r_o}\rangle(\theta,s) = \pm\tfrac{1}{2}(e^{-r_o} \pm \tfrac{1}{2}s)^{-1}$$

This example shows that solutions of the heat equation for Lorentz metrics need not exist for all positive values of the deformation parameter s. Also, in this example one has for $E(s) = E(f^{-}_{r_o s})$

$$\partial_s E(s) = -\tfrac{1}{4}(e^{-r_o} - \tfrac{1}{2}s)^{-2} < 0 \text{ and } \partial_s^2 E(s) = -\tfrac{1}{4}(e^{-r_o} - \tfrac{1}{2}s)^{-3} < 0.$$

B. Let M be the manifold $\mathbb{R}_+ \times S^1$ with the Lorentz metric $ds^2 = dr^2 - r^a d\phi^2, a \in \mathbb{R} \setminus \{0\}$. The equation

$$\left.\begin{aligned} \partial_s\rho_{r_o} &= \frac{a}{2}\,\rho_{r_o}^{a-1} \\ \rho_{r_o}(0) &= r_o \end{aligned}\right\}$$

has the solution

$$\rho_{r_o}(s) = \begin{cases} (r_o^{2-a} + \frac{1}{2}a(2-a)s)^{\frac{1}{2-a}} & a \neq 2 \\ r_o e^s & a = 2 \end{cases}$$

For $a < 0$ the solution exists for $s \in I^a = (-\infty, -\dfrac{2r_o^{2-a}}{a(2-a)})$ and $\rho_{r_o} \to 0$ as $s \to \dfrac{-2r_o^{2-a}}{a(2-a)}$ and $\lim_{s\to-\infty} \rho_{r_o}(s) = \infty$.

(For $a = 0$ the map $(\theta,s) \to (r_o,\theta)$ is a solution of the heat equation). For $0 < a < 2$ the solution exists for $s \in I^a = (-\dfrac{2r_o^{2-a}}{a(2-a)},\infty)$ and

$\rho_{r_s}(s) \to \infty$ as $s \to \infty$, $\rho_{r_o}(s) \to 0$ as $s \to \dfrac{-2r_o^{2-a}}{a(2-a)}$.

For $a = 2$ the solution exists for all values of s and $\lim_{s\to-\infty} \rho_{r_0}(s) = 0$, $\lim_{s\to\infty} \rho_{r_0}(s) = \infty$.

For $a > 2$ the solution exists for $s \in I^a = (-\infty, \frac{-2r_0^{2-a}}{a(2-a)})$ and $\lim_{s\to-\infty} \rho_{r_0}(s) = 0$ and $\rho_{r_0}(s) \to \infty$ as $s \to \frac{-2r_0^{2-a}}{a(2-a)}$.

C. Let M be the manifold $\mathbb{R} \times S^1$ with the metric given by $ds^2 = dr^2 + hd\phi^2$ where $h = \pm(2 + \cos r)$. Suppose $r_0 \in (-\pi,0) \cup (0,\pi)$. The equations

$$\left.\begin{aligned} \partial_s\rho^{\pm}_{r_0} &= \pm \tfrac{1}{2} \sin \rho^{\pm}_{r_0} \\ \rho_{r_0}(0) &= r_0 \end{aligned}\right\}$$

have the solutions $\rho^{\pm}_{r_0}(s) = 2 \arctan [e^{\pm\frac{1}{2}s} \tan r_0/2]$ which exist for all values of the deformation parameter s.

For $r_0 \in (-\pi,0)$ $\rho^{-}_{r_0}(s) \to -\pi$ and $\rho^{+}_{r_0}(s) \to 0$ as $s \to -\infty$

$\rho^{-}_{r_0}(s) \to 0$ and $\rho^{+}_{r_0}(s) \to -\pi$ as $s \to \infty$

For $r_0 \in (0,\pi)$, $\rho^{-}_{r_0}(s) \to \pi$ and $\rho^{+}_{r_0}(s) \to 0$ as $s \to -\infty$

$\rho^{-}_{r_0}(s) \to 0$ and $\rho^{+}_{r_0}(s) \to \pi$ as $s \to \infty$

The curves $S^1 \to M$, $\theta \to (-\pi,\theta)$, $\theta \to (0,\theta)$, $\theta \to (\pi,\theta)$ are geodesics for both metrics.

II. Again let $M = \mathbb{R} \times S^1$ with a Lorentz metric $ds^2 = dr^2 + hd\phi^2$ where $h : \mathbb{R} \to \mathbb{R}_-$ is smooth. Suppose $(r(\theta,s),\phi(\theta,s))$ is a timelike solution of the heat equation satisfying the following: there is an s_0 s.t. $r(\theta,s_0)$ maps into an interval (a,b) where h is strictly decreasing with a local minimum at b. Then the $r(\theta,s)$ map into the same interval for all subsequent s for which it is defined and if the solution is defined for all subsequent s then $r(\theta,s) \to b$ as $s \to \infty$, uniformly in θ.

<u>Proof</u>

$r(\theta,s)$ satisfies the equation

$$\partial_\theta{}^2 r - \frac{1}{2} h'(\partial_\theta \phi)^2 = \partial_s r$$

and $h' < 0$ on (a,b), $h'(b) = 0$. Suppose $r(\cdot,s)$ maps into (a,b) and that $\min_{\theta \in S^1} (r(\cdot,s))$ occurs at $\theta = \theta_0$, then $\partial_\theta{}^2 r(\cdot,s)|_{\theta=\theta_0} \geq 0$ and consequently $\partial_s r > 0$. [There exists a constant $c > 0$ s.t. $(\partial_\theta \phi)^2 > 0$]. Similarly if there existed an s such that $\max_{\theta \in S^1} r(\cdot,s) = b$ occurred at $\theta = \theta_0$ then $\partial_\theta^2(r(\theta,s)) \leq 0$ and therefore $\partial_s r(\theta_0,s) \leq 0$. [Remark: the curves $S^1 \to M$, $\theta \to (b,n\theta)$ $n \in \mathbb{Z}$ are geodesics].

III. A. Let $M = \mathbb{R}^2 \times S^1$ parametrized by $x,y \in \mathbb{R}$ and $\phi \in S^1$. Consider the two metric tensors g_R and g_L on M determined by $ds_R^2 = dx^2 + dy^2 + d\phi^2$ and $ds_L^2 = dx^2 + dy^2 - d\phi^2$ respectively. These two metrics have the same Levi-Civita connection and therefore determine the same heat equation. Let $a \in \mathbb{R}$, $|a| < 1$ and $n \in \mathbb{N}$. Define $f_0 : S^1 \to M$ by $f_0(\theta) = (\frac{a}{n} \cos n\theta, \frac{a}{n} \sin n\theta, \theta)$.

Write f' for $\partial_\theta f$. One has $\langle f_0', f_0'\rangle_L = a^2 - 1 < 0$, f_0 is therefore a timelike curve. Also $\langle f_0', f_0'\rangle_R = 1 + a^2$. $\langle f_0'', f_0''\rangle_L = \langle f_0'', f_0''\rangle_R \equiv a^2n^2$. $f_s : S^1 \to M$ given by $f_s(\theta) = (e^{-n^2s}\frac{a}{n}\cos n\theta, e^{-n^2s}\frac{a}{n}\sin n\theta, \theta)$ is a solution of the heat equation common to both metrics, which coincides with the given f_0 at $s = 0$. As $s \to \infty$ f_s converges uniformly to the curve $\theta \to (0,0,\theta)$ which is a geodesic for both metrics. In this example one has that $\langle f_s', f_s'\rangle_R = 1 + e^{-2n^2s}a^2$, $\langle f_s', f_s'\rangle_L = e^{-2n^2s}a^2 - 1$ are independent of θ. $L_R(s) = \int_0^{2\pi} \langle f_s', f_s'\rangle_R^{\frac{1}{2}} d\theta$ decreases as s increases and $L_L(s) = \int_0^{2\pi} (-\langle f_s'.f_s'\rangle)^{\frac{1}{2}} d\theta$ increases as s increases. By taking a close to zero and n very large one sees that one can choose f_0 and the energies (as defined by g_R and g_L) of f_0 arbitrarily close to their limits but still having the energies changing at an arbitrarily fast rate.

B. Again with (M, g_L) from A, the map $f : S^1 \times [0,\infty) \to M$ given by $f_s(\theta) = (e^{-n^2s}\cos n\theta, \ e^{-n^2s}\sin n\theta, \theta)$ $n > 1$ gives an example of a solution of the heat equation with f_0 spacelike which converges to a timelike geodesic.

IV. Suppose f_0 is a closed C^1 pregeodesic on M either a Lorentz or a Riemannian manifold (in fact this example is the same for any manifold M with a connection ∇). Then $f_0 = f \circ h_0$ where f is a geodesic, and h_0 a C^1 map from S^1 onto S^1. For any curve g one has $\partial_\theta(g \circ h)(\theta) = (\partial_{\theta'} g(h(\theta)))\partial_\theta h(\theta)$ and $\nabla_{\partial_\theta}\partial_\theta(g \circ h) = (\partial_{\theta'} g(h(\theta)))\cdot(\partial_\theta^2 h(\theta)) + (\nabla_{\partial_{\theta'}}\partial_{\theta'} g|_{\theta' = h(\theta)})(\partial_\theta h(\theta))^2$ and there follows for $f \circ h$ that $\nabla_{\partial_\theta}\partial_\theta(f \circ h) = (\partial_\theta^2 h(\theta))\cdot(\partial_{\theta'} f(h(\theta)))$. Now suppose

$h_s : S^1 \to S^1$ is a solution of the equation $\partial_\theta^2 h_s - \partial_s h_s = 0$ for $s \in [0,\infty)$ which is continuous at $s = 0$ and which coincides with h_o at $s = 0$. Then the map $f_s = f \circ h_s$ is a solution of the heat equation, continuous at $s = 0$ and coincides with the given f_o at $s = 0$. $\partial_\theta h_s$ converges to a constant as $s \to \infty$ and f_s converges therefore to a geodesic. For f timelike one has $E(f_s) = E(s) < 0$ and $\partial_s E(s) > 0$, for f lightlike one has $E(s) = \partial_s E(s) \equiv 0$ and when f is spacelike or M Riemannian $E(s) > 0$ and $\partial_s E(s) < 0$.

6. GLOBAL EQUATIONS

The derivation in [ES, Section 7] of global equations replacing the equations (2.1) and (4.1) works in the same way when dealing with Lorentz metrics as for Riemannian metrics. For the sake of clarity a few remarks will be included here.

I. Construction of a metric on a tubular neighbourhood.

Let M be smoothly and properly embedded onto a submanifold M' of some Euclidean space $\mathbf{R}^q$ by the map $w : M \to M' \subset \mathbf{R}^q$. Let N be a tubular neighbourhood of M' constructed using the Euclidean structure on $\mathbf{R}^q$ (denoted by (,)) and let $\pi: N \to M'$ be the projection map. For the Riemannian case it is shown in [ES] that one can construct a smooth Riemannian metric g' on N which has the following properties:

(a) $g(X,Y) = g'(dwX,dwY)$ for all $X,Y \in T_pM$, $p \in M$

(b) For every $u \in \mathbf{R}^q$ such that $u \perp T_xM'$ (w.r.t.(,)) and for $v \in T_xM'$ one has $g'(u,u) = (u,u)$ and $g'(u,v) = 0$

(c) The metric is defined on M' and then translated to any point $x \in N$ along the straight line segment from $\pi(x)$ to x.

For $x \in M'$ define g' on $\mathbf{R}^q(x)$ as follows: let $P_x:\mathbf{R}^q \to T_xM'$ be the orthogonal projection (w.r.t. (,)). For $u,v \in \mathbf{R}^q(x)$ put

$$g'(u,v) = (u,v) - (P_xu,P_xv) + g(dw^{-1}(P_xu),dw^{-1}(P_xv)).$$

With this definition it is obvious that (a) and (b) hold and also that if g is a Lorentz metric then g' is also a Lorentz metric.

II. w considered as a map into (M',g') is an isometry. For isometries one has $dw(\nabla_X Y) = \nabla'_{dwX}\, dWY$ (∇' Levi-Civita connection on M') [O'N p. 90]. Let $f_s : S^1 \to M$ be a smooth map for s in some interval $I \subset \mathbb{R}$. Then one has

$$dw(\nabla_{\partial_\theta} \partial_\theta f_s - \partial_s f_s) = \nabla'_{\partial_\theta}\, dw(\partial_\theta f_s) - dw(\partial_s f_s) = \nabla'_{\partial_\theta}\, \partial_\theta (w \circ f_s) - \partial_s (w \circ f_s)$$

and the conclusion is that $f_s : S^1 \to M$ satisfies the heat equation on M if and only if $w \circ f_s : S^1 \to M'$ satisfies the heat equation on M'.

[O'N, p. 100]. Since (M',g') is a semi-Riemannian submanifold of (N,g') one has

$\bar{\nabla}'_X Y = \nabla'_X Y + II(X,Y)$ for all vector fields X, Y on M' where $\bar{\nabla}'$ (resp. ∇') is the Levi-Civita connection on (N,g') (resp. (M',g')) and II is the shape tensor (or second fundamental form tensor) of $M' \subset N$. II is bilinear over smooth functions, symmetric and $II(X,Y)$ is normal to M' for all vector fields X, Y on M'.

For $f_s : S^1 \to M'$ one has then

$$\bar{\nabla}'_{\partial_\theta} \partial_\theta f_s = \nabla'_{\partial_\theta} \partial_\theta f_s + II(\partial_\theta f_s, \partial_\theta f_s)$$

and

$$\bar{\nabla}'_{\partial_\theta} \partial_\theta f_s - \partial_s f_s = \nabla'_{\partial_\theta} \partial_\theta f_s - \partial_s f_s + II(\partial_\theta f_s, \partial_\theta f_s)$$

and, since $\nabla'_{\partial_\theta} \partial_\theta f_s - \partial_s f_s$ is always tangent to M', it is seen from this that

$$\nabla'_{\partial_\theta} \partial_\theta f_s - \partial_s f_s = 0 \quad \text{if and only if}$$

$$\bar{\nabla}'_{\partial_\theta} \partial_\theta f_s - \partial_s f_s \quad \text{is normal to } M' \text{ and then}$$

$$\bar{\nabla}'_{\partial_\theta} \partial_\theta f_s - \partial_s f_s = II(\partial_\theta f_s, \partial_\theta f_s)$$

The calculations on pp. 139-140 in [ES] for Riemannian metrics remain the same when the metric is Lorentzian. There follows therefore in the same way (see Part I) :

(a) A map $f : S^1 \to M$ satisfies (2.1) if and only if the composition $W = w \circ f$ satisfies

$$\partial_\theta^2 \; W^c = \pi^c_{ab} \; \partial_\theta W^a \partial_\theta W^b \tag{6.1}$$

(b) A deformation $f_s : S^1 \to M$ $(s_0 < s < s_1)$ satisfies (4.1) if and only if $W_s = w \circ f_s$ satisfies

$$\partial_\theta^2 \, W_s^c - \partial_s W_s^c = \pi^c_{ab} \; \partial_\theta W_s^a \; \partial_\theta W_s^b \tag{6.2}$$

(c) Given a smooth $W_s : S^1 \to N$ satisfying (6.2) for $s_0 \le s < s_1$, if W_{s_0} maps S^1 into M' then so does every W_s for $s_0 \le s < s_1$.

7. DERIVATIVE BOUNDS

Let $f_s : S^1 \to M$, $s \in [0, s_1)$ denote a solution of the heat equation on a Lorentz manifold $(M, \langle\ ,\ \rangle)$

I. For the first θ-derivative there is a result in the following special case.

Proposition 7A

Suppose f_s is a timelike solution and that there exists a parallel unit timelike vector field on M. Then $\partial_\theta f_s$ is bounded in the metric $\langle\ ,\ \rangle_\gamma$ determined by γ (Lemma 3B).

Proof

By Proposition 3C the Lorentz metric $\langle\ ,\ \rangle$ and the Riemannian metric $\langle\ ,\ \rangle_\gamma$ have the same Levi-Civita connection. They therefore determine the same heat equation. Therefore the result follows by Theorem 6A of Part I.

This can also be proved directly as follows. W.l.o.g. one can suppose that the curves $f_s : S^1 \to M$ are future directed w.r.t. the time orientation given by γ, for all s. By (3.3) one has $\langle \partial_\theta f_s, \partial_\theta f_s \rangle_\gamma^{\frac{1}{2}} \leq -\sqrt{2}\, \langle \partial_\theta f_s, \gamma \rangle$. It therefore suffices to prove that $-\langle \partial_\theta f_s, \gamma \rangle$ is bounded. Now, for any vector field γ on M one has

$$(\partial_\theta^2 - \partial_s)(-\langle \partial_\theta f_s, \gamma \rangle) = \langle \partial_\theta f_s, \nabla_{\partial_s} \gamma \rangle - 2\langle \nabla_{\partial_\theta} \partial_\theta f_s, \nabla_{\partial_\theta} \gamma \rangle - \langle \partial_\theta f_s, \nabla_{\partial_\theta} \nabla_{\partial_\theta} \gamma \rangle .$$

When γ is parallel the right hand side vanishes. In that case, if for a fixed s $\max_{\theta \in S^1} (-\langle \partial_\theta f_s, \gamma \rangle)$ occurs at $\theta = \theta_0$, one has $\partial_\theta^2(-\langle \partial_\theta f_s, \gamma \rangle)|_{\theta=\theta_0} \leq 0$

and therefore $\partial_s(-\langle\partial_\theta f_s,\gamma\rangle)|_{\theta=\theta_0} \leq 0$. □

II. Let W_s denote the solution of (6.2) corresponding to f_s. The estimate for the second order space derivatives in the case of a Riemannian manifold (M, < , >) proved in ([ES] Section 9) has the following form in the case of domain S^1:

Suppose $|\pi^c_{ab}| \leq C_0, |\partial\pi^c_{ab}/\partial w^d| < C_0$ on M' and

$$A_1 ds_0^2 \leq ds'^2 \leq A_2 ds_0^2$$

where C_0, A_1, A_2 denote positive constants and where ds_0^2 denotes the line element induced on M by the usual metric in $\mathbf{R}^q$. Suppose further that M has non-positive Riemannian curvature and that the energy density is continuous at s = 0. Then given $\varepsilon > 0$ there is a constant C independent of the solution f_s such that

$$|\partial_\theta^2 W_s^c(\theta)| < C\bar{e}(f_0)[1 + \bar{e}(f_0)^{\frac{1}{2}} + \sup\{|W_0^c(\theta)| : \theta \in S^1\}]$$

for $t \geq \varepsilon$ where $\bar{e}(f_0) = \sup\{\frac{1}{2}\langle\partial_\theta f_0(\theta),\partial_\theta f_0(\theta)\rangle : \theta \in S^1\}$.

Let $\|\ \|$ denote the usual norm on $\mathbf{R}^q$ and put

$$m(f) = \sup\ \{\|\partial_\theta W_s(\theta)\|^2 : \theta \in S^1, s \in [0,s_1)\}.$$

An inspection of the proof of the result above for Riemannian manifolds reveals that if one uses m(f) rather than the energy density, it will work in the same way to give the following result for a Lorentz manifold.

Theorem 7 B

Suppose there is a positive constant C_o such that $|\pi^c_{ab}| \leq C_o$ and $|\partial\pi^c_{ab}/\partial w^d| \leq C_o$ on the image of W_s and that $m(f) < \infty$, then there exists a constant C such that

$$|\partial^2_\theta W^c_s(\theta)| \leq C \quad \text{for all } \theta, s \text{ for which the solution is defined.}$$

<u>Theorem 7 B</u>

Suppose there is a positive constant C_o such that $|\pi^c_{ab}| \leq C_o$ and $|\partial\pi^c_{ab}/\partial w^d| \leq C_o$ on the image of W_s and that $m(f) < \infty$, then there exists a constant C such that

$$|\partial^2_\theta W^c_s(\theta)| \leq C \quad \text{for all } \theta, s \text{ for which the solution is defined.}$$

8. EXISTENCE AND UNIQUENESS OF SOLUTIONS

I. In [ES] the following theorem is proved for solutions of heat equations on Riemannian manifolds (Theorem 10A):

Theorem 8A

Let f_s and f'_s be two solutions of the heat equation, both continuous along with their first order θ-derivatives at $s = 0$. If $f_o = f'_o$ then the two solutions coincide for all (relevant) $s > 0$.

This theorem is also true for Lorentz manifolds. The only change that needs to be made in the proof is to use

$$m(f_s) = \sup \{ \| \partial_\theta W_s(\theta) \|^2 : \theta \in S^1 \}$$

instead of

$$\bar{e}(f_s) = \sup \{ \tfrac{1}{2} \langle \partial_\theta f_s(\theta), \partial_\theta f_s(\theta) \rangle : \theta \in S^1 \}$$

where, as before, W_s is the solution of (6.2) corresponding to f_s.

II. The proofs of Theorems 10B and 10C in [ES] can be used to prove the following analogues in the case of Lorentz metrics.

Theorem 8B

Let K be a compact subset of M. Suppose $f : S^1 \to M$ is a continuously differentiable curve with image in K. Put $W_o = w \circ f$ and $m(f) = \sup \{ \| \partial_\theta W_o(\theta) \|^2 : \theta \in S^1 \}$. Then there is a positive constant s_1 depending only on K and m(f) such that (4.1) has a solution f_s for $0 \leq s \leq s_1$, which is continuous at $s = 0$ along with its θ-derivative and which coincides with f at $s = 0$.

<u>Theorem 8C</u>

Let M satisfy the embedding conditions of Theorem 7B and suppose it is complete in the Riemannian metric induced by the usual metric on $\mathbf{R}^q$. Suppose further that $f : S^1 \to M$ is a continuously differentiable curve such that there is a constant C such that for any solution f_s of (4.1) (and W_s the corresponding solution of (6.2)) which is continuous along with its θ derivative at $s = 0$ and which coincides with f at $s = 0$ one has

$$m(f_s) = \sup \{ \| \partial_\theta W_s \| : \theta \in S^1 \} < C \text{ for all } s.$$

Then any such solution can be extended (uniquely) to be defined for all $s \geq 0$.

9. SUBCONVERGENCE OF TIMELIKE SOLUTIONS

In this section let $f_s : S^1 \to M$, $s \in [0,\infty)$ denote a timelike solution of the heat equation on the Lorentz manifold $(M, \langle\ ,\ \rangle)$ such that the image of f is contained in a compact $K \subset M$. Assume further that the tangent fields $\partial_\theta f_s$ are uniformly bounded; i.e. if g_R is a Riemannian metric on M then there is a constant C s.t. $g_R(\partial_\theta f_s, \partial_\theta f_s) < C$ for all $\theta \in S^1$, $s \in [0,\infty)$. Since the image of f is contained in a compact subset of M, this property is independent of g_R, in the sense that if such a constant exists for a given g_R then the same is true for every other Riemannian metric on M.

In the following let g_R be a fixed but arbitrary Riemannian metric and write $\|v\|_R = (g_R(v,v))^{\frac{1}{2}}$.

By Proposition 4A the energy density of f is bounded from above. Therefore there exists a constant $C_1 > 0$ such that

$$C_1 \leq \| \partial_\theta f_s(\theta) \|_R \text{ for all } \theta \in S^1,\ s \in [0,\infty) \tag{9.1}$$

Also, since $\| \partial_\theta f_s \|_R$ is bounded from above there exists a constant C_2 s.t.

$$C_2 \leq \langle \partial_\theta f_s(\theta), \partial_\theta f_s(\theta) \rangle \text{ for all } \theta \in S^1,\ s \in [0,\infty) \tag{9.2}$$

and the angles (as defined by g_R) between $\partial_\theta f_s$ and the light cone and (therefore) between $\partial_\theta f_s$ and $(\partial_\theta f_s)^{\perp}$ ($\perp$ as defined by $\langle\ ,\ \rangle$) are uniformly bounded away from zero.

From (9.2) there follows at once that there is a constant C_3 such that

$$L(s) = L(f_s) = \int_0^{2\pi} (-\langle\partial_\theta f_s(\theta),\partial_\theta f_s(\theta)\rangle)^{\frac{1}{2}} d\theta < C_3$$

$$\text{for all } s \in [0,\infty) \tag{9.3}$$

and

$$E(s) = E(f_s) = \int_0^{2\pi} \langle\partial_\theta f_s(\theta),\partial_\theta f_s(\theta)\rangle d\theta > -C_3$$

$$\text{for all } s \in [0,\infty) \tag{9.4}$$

By Proposition 4C and its corollary

$$\partial_s L(s) = \int_0^{2\pi} (-\langle\partial_\theta f_s,\partial_\theta f_s\rangle)^{-\frac{1}{2}} [\langle\nabla_{\partial_\theta}\partial_\theta f_s,\nabla_{\partial_\theta}\partial_\theta f_s\rangle - \frac{\langle\nabla_{\partial_\theta}\partial_\theta f_s,\partial_\theta f_s\rangle^2}{\langle\partial_\theta f_s,\partial_\theta f_s\rangle}] d\theta \geq 0$$

By (9.3) one has therefore

$$\int_0^{\infty} ds \int_0^{2\pi} (-\langle\partial_\theta f_s,\partial_\theta f_s\rangle)^{-\frac{1}{2}} [\langle\nabla_{\partial_\theta}\partial_\theta f_s,\nabla_{\partial_\theta}\partial_\theta f_s\rangle - \frac{\langle\nabla_{\partial_\theta}\partial_\theta f_s,\partial_\theta f_s\rangle^2}{\langle\partial_\theta f_s,\partial_\theta f_s\rangle}] d\theta < C_3 \tag{9.5}$$

By (9.2) $-\langle\partial_\theta f_s,\partial_\theta f_s\rangle$ is uniformly bounded from above.

Therefore (9.5) implies that there is a constant C_4 such that

$$\int_0^{\infty} ds \int_0^{2\pi} [\langle\nabla_{\partial_\theta}\partial_\theta f_s,\nabla_{\partial_\theta}\partial_\theta f_s\rangle - \frac{\langle\nabla_{\partial_\theta}\partial_\theta f_s,\partial_\theta f_s\rangle^2}{\langle\partial_\theta f_s,\partial_\theta f_s\rangle}] d\theta < C_4 \tag{9.6}$$

Recall (4.6)

$$\partial_s E(s) = \int_0^{2\pi} - \frac{\langle \nabla_{\partial_\theta} \partial_\theta f_s, \partial_\theta f_s \rangle^2}{\langle \partial_\theta f_s, \partial_\theta f_s \rangle} d\theta + \int_0^{2\pi} - \left\langle \nabla_{\partial_\theta} \partial_\theta f_s - \frac{\langle \nabla_{\partial_\theta} \partial_\theta f_s, \partial_\theta f_s \rangle}{\langle \partial_\theta f_s, \partial_\theta f_s \rangle} \partial_\theta f_s, \right.$$

$$\left. \nabla_{\partial_\theta} \partial_\theta f_s - \frac{\langle \nabla_{\partial_\theta} \partial_\theta f_s, \partial_\theta f_s \rangle}{\langle \partial_\theta f_s, \partial_\theta f_s \rangle} \partial_\theta f_s \right\rangle d\theta$$

The integrand in the second integral on the right hand side is the negative of the bracket in the integral in (9.6). By (9.4) one has $-C_3 < E(s) < 0$. The first integral on the right hand side of (4.6) is always non-negative and the second always non-positive. Combining these facts one sees that there is a constant C_5 such that

$$C_5 < \int_0^\infty ds \int_0^{2\pi} \frac{\langle \nabla_{\partial_\theta} \partial_\theta f_s, \partial_\theta f_s \rangle^2}{\langle \partial_\theta f_s, \partial_\theta f_s \rangle} d\theta \tag{9.7}$$

From (9.6) and (9.7) one deduces the

<u>Lemma 9 A</u>

There exists a sequence of s values $s_1, s_2, s_3 \ldots, s_k \to \infty$ such that

$$\int_0^{2\pi} \frac{\langle \nabla_{\partial_\theta} \partial_\theta f_{s_k}, \partial_\theta f_{s_k} \rangle^2}{\langle \partial_\theta f_{s_k}, \partial_\theta f_{s_k} \rangle} d\theta \to 0 \text{ as } k \to \infty \tag{9.8}$$

and

$$\int_0^{2\pi} [\langle \nabla_{\partial_\theta} \partial_\theta f_{s_k}, \nabla_{\partial_\theta} \partial_\theta f_{s_k} \rangle - \frac{\langle \nabla_{\partial_\theta} \partial_\theta f_{s_k}, \partial_\theta f_{s_k} \rangle^2}{\langle \partial_\theta f_{s_k}, \partial_\theta f_{s_k} \rangle}] d\theta \to 0 \text{ as } k \to \infty \qquad (9.9) \quad \square$$

Lemma 9B

For $\partial_s W(\theta, s_k) = dw(\partial_s f(\theta, s_k))$ (s_k from the previous lemma) one has that the $\partial_s W(\theta, s_k)$ converge in the mean to zero as $k \to \infty$.

Proof

Write $\partial_s f_{s_k} = \nabla_{\partial_\theta} \partial_\theta f_{s_k} = \tau^{\shortparallel}_{s_k} + \tau^{\perp}_{s_k}$ where $\tau^{\shortparallel}_{s_k}$ is parallel to $\partial_\theta f_{s_k}$ and $\tau^{\perp}_{s_k}$ orthogonal to it ($\shortparallel$ and $\perp$ referring to $\langle\ ,\ \rangle$). As has been pointed out previously the integral in (9.8) is

$\int_0^{2\pi} \langle \tau^{\shortparallel}_{s_k}, \tau^{\shortparallel}_{s_k} \rangle d\theta$ and the integral in (9.9) is $\int_0^{2\pi} \langle \tau^{\perp}_{s_k}, \tau^{\perp}_{s_k} \rangle d\theta$.

Therefore (9.8) and (9.9) imply

$$\int_0^{2\pi} \|\partial_s f_{s_k}\|_R \, d\theta \to 0 \text{ as } k \to \infty$$

and this in turn gives the result. $\square$

Remark

The identity (7.1) of Part I is still valid in the case of Lorentz manifolds. Also, in the situation under discussion, the last term of that identity is uniformly bounded (see Theorem 7B). However because of the non-definite nature of the metric one cannot conclude from that in the same way as in Part I that the $\partial_s W_s(\theta)$ converge in the mean to zero.

Theorem 9C

There is a sequence $s_1, s_2, s_3 \ldots$ with $s_k \to \infty$ such that the curves $f_k = f_{s_k}$ converge uniformly to a closed geodesic f.

Proof

This is the analogue of Theorem 7A of Part I. The same proof will work except that here one has to start by choosing a sequence such as the one in the above lemmas and then choosing from that a subsequence that converges uniformly. $\square$

APPENDIX

To conclude this thesis, a few unsolved problems in the study of solutions of the heat equation for closed curves. In the case of timelike solutions on Lorentz manifolds, example IB (and other examples) of Section 5 of Part II suggest that the boundedness and the length of the interval of existence depend on the "growth" of the manifold. It may therefore be possible to invent conditions on the manifold (see boundedness conditions in Part I) which will ensure boundedness and/or existence of solutions for all positive values of the deformation parameter. Also, choosing a Riemannian metric on the Lorentz manifold, it is not known whether assuming the boundedness of a solution and an upper bound on the Riemannian length in the t-homotopy class of the initial curve will give a priori bounds on the first θ-derivative of the solution. Finally, the convergence results both for the Riemannian case and for timelike solutions on Lorentz manifolds only give subconvergence of solutions, i.e. existence of a sequence of s-values $s_1, s_2, s_3, \ldots$ such that the f_{s_k} converge uniformly to a closed geodesic as $k \to \infty$. It is, in general, an unsolved problem to prove the convergence of solutions, i.e. that the f_s converge uniformly to a closed geodesic as $s \to \infty$.

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